Synthetic Cash Flow Model with Singularity Functions for Unbalanced Bidding Scenarios

Abstract

Construction contractors may utilize unbalanced markup bidding strategies, e.g. front-loading, to seek more beneficial cash flows, or even protect their target profits. A difference between ethical and unethical unbalanced bidding is whether the bid has been mathematically or even materially modified, whereas the former may be allowable, but the latter should be rejected, as it fails to cover even cost of later activities. Unbalanced bidding adds to the complexity of cash flow models and requires a new model that can calculate different scenarios accurately and efficiently. Basic new theory is explored for unbalanced bidding in cash flow models by employing the range-based class of singularity functions: First, a new synthetic balanced cash flow model that can accurately consider the time value of money (TVM) and retainage is derived to serve as the foundation; second, various unbalanced markup functions including two-phase, linear, and non-linear types are defined both cumulatively and non-cumulatively; third, the performance of the extended synthetic cash flow model for unbalanced bidding is investigated mathematically.

Keywords: Cash flow analysis, cash flow management, bidding, unbalanced bidding, singularity functions
Introduction

“The vital importance of cash flow management to construction companies is widely recognized. Despite the practice of interim payments to the contractor, usually made at monthly intervals, it has widely remained the belief that the contractor tends to act as financier until the later stages of the project” (Green 1989, p. 55). To aid their financial performance, contractors may employ so-called unbalanced bidding on construction projects, i.e. distributing the markup unevenly over the duration of a construction project and thus receiving more money earlier, while being paid the same total amount (Cattell et al. 2007). This practice exists in the construction industry, but is fraught with contentious ethical issues (Kenley 2003). Besides the reason that contractors may seek to become more profitable (through TVM) by unbalancing their bids, surprisingly owners may also cause unbalancing, albeit not deliberately, e.g. an imprecise work quantity estimated by owners in preparing a unit price contract “forces the construction community to unbalance their unit prices to protect their fixed costs and target profit” (Gransberg and Riemer 2009, p. 1141).

Unbalanced bidding comprises three types: Front-loading, back-loading, and individual rate loading. Front-loading means that a bidder modifies the unit price of items that occur early in the schedule by increasing its profit to collect more money earlier. Back-loading, conversely, is increasing the price of later items, e.g. if concerned about a high inflation rate. Individual rate loading is increasing the price on items whose quantity the owner has underestimated (Cattell et al. 2007). From a mathematical view, front- and back-loadings are similar in that a price depends on when the item occurs in the schedule. In other words, price is a function of time. However, for individual rate loading, price is a function of quantity, which – assuming a constant productivity – may be a function of time itself. The scope of this study is limited to exploring mechanisms of
front- and back-loading in unbalanced bidding. More intricate scenarios, such as individual rate loading or combinations of unbalanced bidding types, will be addressed under future research.

4 Literature Review

Prior studies on unbalanced bidding can be categorized by view: From contractors’ perspective, who may be “interested in the optimization of a contractor’s cash flow by unbalancing a bid and how not to be caught in the process” (Arditi and Chotibhongs 2009, p. 727) and from owners’, who wish to detect and prevent unbalanced bids. The latter topic is divided into mathematically versus materially unbalanced bids, whose difference is that the former “contain[ing] lump sum or unit bid items which do not reflect reasonable actual costs plus a reasonable proportionate share of the bidder's anticipated profit, overhead costs, and other indirect cost”, whereas the latter will “generate[s] a reasonable doubt that award to the bidder submitting a mathematically unbalanced bid will result in the lowest ultimate cost to the Federal Government” (23 C.F.R. § 635.102). Their distinction was explained more intuitively as “mathematically unbalanced bid is one in which each bid item (or breakdown of scheduled values in a lump-sum contract) fails to carry its proportionate share of the overhead and profit in addition to the necessary costs for the item” versus that “materially unbalanced [bid] has shifted not only a disproportionate amount of overhead and profit, but also some portion of the actual cost of elements of work” (Manzo 1997, pp. 2-3), so that the contractor performs some line items effectively below cost. Per Federal law, bids that are merely “mathematically unbalanced, but not found to be materially unbalanced, may be awarded” (23 C.F.R. § 635.114), presumably because of a lower risk of financial failure.

It still remains unsolved how an owner can assess if a bid is materially unbalanced. Since contractors experience dissimilar situations that cause them to bid specific prices (i.e. cost plus
profit) for the same item, owners cannot be sure if a bid value is truly abnormal. For example, the reason why one contractor quotes a lower price for earthmoving may be that it has already recovered the depreciation of their owned heavy equipment, while another pays more expensive rental costs. But from the owner’s perspective, the former bid appears dubious. To make up its aberrant but balanced bid, the former contractor may have to add a higher markup to this item to reach a similar price as its competitors, which creates the ironic result that “contractors may need to unbalance their prices to produce a tender that has the appearance of not being unbalanced” (Cattell et al. 2011, p. 1056). Remaining conscious of – but neutral to – the ethical debate, unbalanced bidding is regarded herein merely as a cash flow management strategy that some contractors may use, with the intent to formalize its mechanism and obtain theoretical insights.

Unbalanced Bidding Studies

Unbalanced bidding is understood as two steps for analysis: The first “determines optimum bids in a competitive-bidding situation where each competitor submits one closed bid” (Friedman 1956, p. 104) whose target is that “the markup should be high enough to ensure a profit if won, yet not too high to lose the job” (Yuan 2013, p. CON-058-9). Friedman (1956) provided a mathematical model to calculate the probability of winning, whereas Gates (1967) presented another one and advocates the concept of unbalanced bidding. Their classical debate was “the way of aggregating the individual probabilities for the overall probability of winning” (Yuan 2013, p. CON-058-4). Further studies explored competitor utility theory (Awwad and Ioannou 2012, Cattell et al. 2011) and competitor correlation and its influence on markups (Yuan 2011).

After setting the total bid price in the first step, the second step is unbalancing the markup for profit to maximize its present value (PV), while leaving the total bid price unchanged. Stark
(1968) provided a linear programming model that considered the constraints of a known total bid, relationships between unit prices, and the schedule of payments. However, such constraints were flawed, because of their “upper and lower bounds that are seemingly arbitrarily decided upon” (Cattell et al. 2007, p. 565). Ashley and Teicholz’ (1977) linear unbalancing model multiplied a higher factor with earlier items in the schedule and lowered that factor later. The factor changed linearly with time, so that the total bid price remained unchanged. However, this approach failed to determine how much better the PV became. Also, further scenarios were omitted, e.g. non-linear unbalancing. Diekmann et al. (1982) presented a model to maximize the PV of a bid and simultaneously minimize risk, which was defined as the probability of such PV not surpassing the contractor’s anticipation. However, the optimum markup had to be assumed and determining its value was left unsolved. Cattell (1987) explored a comprehensive formulation for the PV of the contractor’s profit, which considered TVM, retainage off progress payments, and risk. Tong and Lu (1992) built a model for individual rate loading. More recent research broadened the area: Christodoulou (2008) advocated that entropy, which is an index of disorder, should be used in an unbalanced bidding model to reach both maximum PV and minimum financial disorder. Afshar and Amiri (2010) used fuzzy linear programming to model uncertainties in unit prices, quantities, and constraints. Cattell et al. (2011) applied cumulative prospect theory from microeconomics to calculate the bidder’s profit expectation at different risk levels for different pricing options. Only some of the previous researchers considered the TVM in their unbalanced bidding models, and if so, many failed to calculate the amount of higher PV that unbalancing creates. All studies only treated unbalancing between activities; none of them considered unbalancing within activities, although activities may exceed a single period, or – even if often defined as less than two weeks long versus monthly periods (O’Brien and Plotnick 2010) – may span across period boundaries.
A model that can accomplish both would be sufficiently powerful to treat all possible scenarios. Also, the underlying models of previous studies were less accurate and flexible than the synthetic cash flow model that uses so-called singularity functions as explained in the following sections.

Justification and Research Objectives

A gap in the body of knowledge exists in that no prior model could unbalance within activities. Specifically, they limited unbalancing to between work items, i.e. “major construction activities” (Cui et al. 2010, p. 367). But “to forecast any item’s cash outflow should require that one give consideration to the item’s cost breakdown into its constituent components which may be of many different types (labor, materials, subcontractors, etc.), each with substantially different delays/advances between when the item is built and when each of these costs have to be paid for” (Cattell et al. 2007, p. 569). A need thus exists to derive a synthetic model that connects unbalanced bidding concepts with scheduling, which will enable ‘what-if’ experimentation. Such an unbalanced bidding model should be developed at a detailed level of activities to be able to reflect any schedule changes. For brevity of the discussion, the new model is presented here for various scenarios of unbalancing within activities; while it is certainly theoretically possible to handle the aforementioned types of specific costs by defining ‘cost-specific sub-activities’ for labor, materials, and equipment, details on if and how to unbalance between such types of costs are left to be explored in future research. Three Research Objectives are set to be accomplished:

1. Equipping synthetic cash flow model with retainage to first examine balanced bidding;
2. Expressing the varying markup in unbalanced bidding by way of singularity functions;
3. Expanding synthetic cash flow models toward diverse scenarios of unbalanced bidding;
Practical Meanings of Research Objectives

By addressing Research Objective 1, managers will receive a comprehensive and detailed model against which to compare in their decision-making processes; based on Research Objective 2, managers will be able to decide what approach for unbalancing a markup to use; Objective 3 will enable managers to maximize profit that can be realized under possible scenarios of their projects.

Derivation of the Model

The distributive law (adding variable values then multiplying the sum with a factor is equivalent to summing individual products of a factor with each variable) is applied to model the cash flow of individual activities within an entire project. As is common in studying cash flow phenomena, it is assumed that cost is evenly distributed over the activity duration (Elazouni and Metwally 2005). However, this for brevity of the formulation, whereas the particular type of mathematical functions that are employed and explained in the following sections could express uneven cost.

By modeling the cost, bill, payment, interest, retainage, and balance for individual activities, then adding their effects per Equation 1, one can model a project with numerous activities and easily make comparisons if individual activities change. Time parameters (duration, shift, and delay) will be included in this model; it thus integrates cash flow with scheduling. Since time $y$ is the independent variable within the continuous model, at any time the balance of all cash flows can be calculated, e.g. at the last pay time (i.e. project finish plus one bill-to-payment delay period).

$$z(y)_{project} = \sum z(y)_{activity}$$  \hspace{1cm} (1)
**Terminology**

The term ‘synthetic’ will be used to indicate that the new cash flow model is synthesized from the aforementioned components of cost, bill, payment, interest, retainage, and balance under a more efficient modeling paradigm (using the distributive law) than was traditionally employed. Traditional models used a project balance perspective that arduously calculated a present value contribution for the balance at each period end only, or a slightly more efficient investment pool perspective that summed inflows and outflows separately (Park 2011), but still only functioned for the discrete time points of the project start (present value) or project finish (future values).

The new model will consider TVM, periodic interest and payment (via signal functions), and withholding retainage and releasing it later. Modeling the future value of cost and payment before taking their difference gives the same balance as the chronological approach that has been widely used, including by these authors (Reference 3). Such traditional models had to resort to defining discrete balance, interest, and payment ‘events’ at each period end, which are separated by an (implicitly assumed) small offset $\varepsilon$ (to obtain three different values at one integer time).

**Structure of the Model**

The flowchart of Figure 1 illustrates the structure of the model: Per the literature review, a gap in the body of knowledge exists regarding an unbalanced bidding model within activities, nor has any such model been integrated with scheduling. The synthetic cash flow model is built from singularity functions, the same mathematical expressions that have already been applied to linear schedules. They will also provide signal functions that are needed for periodic phenomena. The central concept of treating the markup as a function of time is developed. A constant markup for the simple case of balanced bidding serves as the baseline for comparison (Research Objective}
1), while variable markup functions are further categorized into three profiles, two-phase, linear, and non-linear, which are considered under mathematically and materially unbalanced bidding scenarios as defined by the Federal government (Research Objective 2). Together, they form the newly derived synthetic cash flow model for unbalanced bidding (Research Objective 3).

<Insert Figure 1 here>

Singularity Functions Definition

Equation 2 denotes the mathematical expression for the basic term of singularity functions. It uses \( y \) (time) as the independent variable because \( x \) has been used to represent work quantity when modeling linear scheduling problems (Reference 2). The dependent variable \( z \) (cost) changes according to the relative size of \( y \) and the cutoff \( a \): Thus \( z \) is equal to zero if \( y \) is smaller than \( a \), otherwise the pointed brackets are evaluated as round brackets by obeying normal algebra rules. The other two parameters that are located outside brackets influence the behavior of the function. The strength \( s \) acts to amplify the value depending on the exponent \( n \); it is the slope if \( n = 1 \), or the intercept if \( n = 0 \). Traditionally, singularity functions have been stated in a right-continuous format, but are certainly also allowed to be defined as left-continuous if necessary.

Singularity functions were first written by A. O. Föppl (1854-1924) and W. H. Macaulay (1853-1936) to calculate structural elements under different loads (Macaulay 1919). Their merit is that by adding terms, these various loads are actually superposed on “[c]omputed ranges [that] are defined among any locations where any variable changes, e.g., its loads, beam cross-section, or support conditions” (Reference 3, p. 524). Cash flow analysis can benefit from such analogy-based research, because “[t]he cumulative interaction of outflows … and inflows … creates a
profile with a complex zigzag shape” (Reference 3, p. 523). In recent work by the second author, singularity functions succeeded in modeling financial phenomena (Reference 3, Reference 4).

\[
z(y) = s \cdot (y - a)^n = \begin{cases} 
0 & \text{for } y < a \\
 s \cdot (y - a)^n & \text{for } y \geq a
\end{cases}
\] (2)

Note that the exponent in Equation 2 controls whether or not a behavior is cumulative, which is explained in a following section. Generally, the difference is merely that if a cumulative function has an exponent \( n \), then its non-cumulative form is the first order derivative with \( n - 1 \).

8 Activity Equation for Linear Schedule

Linear and repetitive scheduling techniques are advantageous for expressing changes within the activity productivity while obeying sequencing relations and other constraints between activities (Reference 2). They can be mathematically expressed with singularity functions per Equation 4, where \( x \) represents the work as a function of time \( y \), \( U \) is the total work quantity of the activity, which divided by its duration \( (D + d_2) \) is the productivity. Schedule changes to planned activity start \( a_s \) or finish \( a_f \) are indicated via a shift \( d_1 \) that moves the start or a delay \( d_2 \) that extends the duration, so that the actual dates become \( a_s' = a_s + d_1 \) and \( a_f' = a_f + d_1 + d_2 \). Activity progress is typically assumed as linear, but could also be non-linear if necessary to fit actual data. Details of the critical path analysis have been explained elsewhere (Reference 2). Briefly speaking, each of the activities and buffers are written per Equation 3, placed sequentially into initial positions in a schedule, and overlapped as far as constraints allow to achieve the minimum project duration.

\[
x(y) = \frac{U}{D + d_2} \cdot \left[ (y - a_s')' - (y - a_f')' \right]
\] (3)
Existing Cash Flow Models with Singularity Functions

Cash flow modeling requires accurate and efficient calculation. The aforementioned traditional approach had to calculate balances at period ends with no less than three steps: First, calculating the balance just before charging interest, then after an infinitesimal time offset $\varepsilon$ subtracting the interest, and after another $\varepsilon$ finally adding any received payment. Previous studies applied this stepwise approach either implicitly within their calculations (e.g. Halpin and Woodhead 1998) or explicitly discussed it (Reference 6). While it returned correct balances, any values throughout the period itself remained unknown. Such iterative chronological approach was arduous and disjointed. It would provide a serious limitation to a cash flow model that seeks to properly include financing; fortunately singularity functions can offer a continuous alternative.

Synthetic Cash Flow Model with Balanced Markup

Reference 8 sets up a synthetic cash flow model per Equations 4 to 16 with three special items: A signal function to control periodicity, a pay function that can consider the retainage and TVM, and a cost function with interest. Inputs are the total cost of each activity $C$, the bill period $p$, bill-to-pay-delay $b$, retainage $r$, balanced distributed markup $M$, and financing interest $i$ on balances.

Signal Function

Many cash flows are periodic, e.g. receiving payments or changing interest. Signal functions are a type of singularity functions that incorporate a roundup operator $\lceil \rceil$ or rounddown operator $\lfloor \rfloor$ to round the operand to the nearest integer. Taking their difference will generate a periodic signal (Reference 8). For example, $(\lfloor y \rfloor - a) - (\lceil y \rceil - (a + 1))^1$ is an infinite signal function, whose first term is active from $a$ onward with a stepped profile; the second one is stepped from $a + 0.000...1$
onward. Subtracting it yields a series of signals at every integer from $\lceil a \rceil$ onward, and is otherwise zero (Reference 8). This feature allows modeling cash flows efficiently: Multiplying a continuous function with a signal function yields a periodic sampling, which is useful to generate a pay function. Equations 4 and 5 provide signal functions for pay and charging interest. Note the pay signal depends on both the bill period $p$ and the bill-to-pay-delay $b$, where $p$ denotes how frequently a contractor sends a bill to the owner (e.g. monthly) and $b$ is the wait between requesting pay and receiving it. Thus $p$ determines the cycle time of pay, and $b$ sets the first pay time (Reference 8). By dividing each term in Equation 4 by $p$, the period can be any real number. Subtracting $b$ from $y$ shifts the signal by $b$ time units. As interest is typically charged periodically by the bank, the parameters $b$ and $p$ are generally not needed in Equation 5. The signal functions will be used in the following sections to control when to initiate payments or charge interest.

$$z_{\text{pay\_signal}}(y) = \left[ \left( \frac{y-b}{p} \right) - \left( \frac{a_s}{p} \right) \right] \left[ \left( \frac{y-b}{p} \right) - \left( \frac{a_r}{p} + 1 \right) \right]$$

$$z_{\text{int\_signal}}(y) = \left[ (y-a_s) \right] - \left[ (y-a_r + 1) \right]$$

Pay Function

The aforementioned periodic sampling via singularity functions is applied to yield Equation 6 for each pay. Multiplying the bill slope $C \cdot (1 + M) / (D + d_2)$ with $p$ and the pay signal $z_{\text{pay\_signal}}$ returns the pay amount at pay time and zero at non-pay time. Applying the retainage $r$ calculates the retained amount for each payment per Equation 7. At the last pay time, the accumulated total retainage must be released. Equation 8 thus uses a controller term $\left( \frac{(y-b)}{p} - \left( \frac{a_r^{\text{project}}}{p} \right) \right)^6$ to check if it is the last pay time. It continues to subtract retainage at each pay time, but releases the
retainage $\sum z_{ret}(y)$ once the last pay time occurs. Equation 9 is the last pay time, considering whether the activity finish $a_f^*$ is divisible by $p$ (Reference 8). Equation 10 incorporates the TVM for each pay amount by applying the different compounding factors accordingly to periodic pay.

$$z_{each\_pay}(y) = \frac{C \cdot (1 + M)}{D + d_2} \cdot p \cdot z_{pay\_signal}$$ (6)

$$z_{ret}(y) = r \cdot z_{each\_pay}(y)$$ (7)

$$z_{pay\_with\_ret}(y) = z_{pay\_with\_ret}(y) - z_{ret}(y) + \left(\frac{y - b}{p} - \frac{a_f^*\_project}{p}\right)^n \cdot z_{pay\_signal} \cdot \sum z_{ret}(y)$$ (8)

$$y_{last\_pay} = \left(\left\lceil \frac{a_f^*}{p}\right\rceil \cdot \left\lfloor \frac{a_f^*}{p} \right\rfloor \right) \cdot (a_f^* + b) + \left(1 - \left(\left\lceil \frac{a_f^*}{p} \right\rceil - \left\lfloor \frac{a_f^*}{p} \right\rfloor \right)\right) \cdot \left(\left\lceil \frac{a_f^*}{p} \right\rceil - \frac{a_f^*}{p}\right) \cdot p$$ (9)

$$z_{PF\_pay}(y) = z_{pay\_with\_ret}(\left\lceil \frac{a_s^*}{p} \right\rceil + b + p) \cdot (1 + i)^{\left\lceil \frac{y - b}{p} \right\rceil} \cdot \left(y - \left(\left\lceil \frac{a_s^*}{p} \right\rceil + b + p\right)\right)^n$$

$$+ z_{pay\_with\_ret}(\left\lceil \frac{a_s^*}{p} \right\rceil + b + 2p) \cdot (1 + i)^{\left\lceil \frac{y - b}{p} \right\rceil} \cdot \left(y - \left(\left\lceil \frac{a_s^*}{p} \right\rceil + b + 2p\right)\right)^n$$

$$+ z_{pay\_with\_ret}(\left\lceil \frac{a_s^*}{p} \right\rceil + b + 3p) \cdot (1 + i)^{\left\lceil \frac{y - b}{p} \right\rceil} \cdot \left(y - \left(\left\lceil \frac{a_s^*}{p} \right\rceil + b + 3p\right)\right)^n$$

$$\ldots + z_{pay\_with\_ret}(y_{last\_pay}) \cdot (1 + i)^{\left\lfloor \frac{y - y_{last\_pay}}{p} \right\rfloor} \cdot \left(y - y_{last\_pay}\right)^n$$ (10)

**Cost Function**

Interest is calculated by compounding the principal to the end of each period, which yields the amount that is charged. Equation 11 is the cost with interest function that allows even fractional start or finish caused by the shift $d_1$ and delay $d_2$, if any. Exact interest for linearly growing cost is calculated here, with three possible scenarios: A balance may grow in the first half of a period, the second, or across the entire period (Reference 4). Cash flow without TVM can be calculated.
per Equations 12 to 14. Equation 13 is the bill function that multiplies the markup factor \((1 + M)\) with the cost function. Equation 14 is a step cost function with the aforementioned rounddown operator, which only returns a cost value at the end of each period, but is unchanged during it. Subtracting the step cost from the cost with interest is the interest at signal function per Equation 15, which returns the interest at the end of each period. Finally, subtracting the combined effect of the cost without TVM (because it is already included in \(z_{FV_{pay}}\) and \(z_{int_{at_{signal}}}\)) and the interest at signal function from the future value of pay function is the balance function per Equation 16.

\[
z_{cost_{cumu}}(y) = \frac{C}{D + d_2} \left[ (y - a_S^i)^{t_{int_{signal}}(a_S^i)} - (y - a_F^i)^{t_{int_{signal}}(a_F^i)} \right]
\]

\[
z_{bill_{cumu}}(y) = (1 + M) \cdot z_{cost_{cumu}}(y)
\]

\[
z_{step_{cost}}(y) = \frac{C}{D + d_2} \left[ (y - a_S^i)^{t_{int_{signal}}(a_S^i)} - (y - a_F^i)^{t_{int_{signal}}(a_F^i)} \right]
\]

\[
z_{int_{at_{signal}}}(y) = z_{cost_{int}}(y) - z_{step_{cost}}(y)
\]

\[
z_{balance}(y) = z_{FV_{pay}}(y) - z_{cost_{cumu}}(y) - z_{int_{at_{signal}}}(y)
\]

\[
Verification
\]

For Example 1, let \(C = 500,000\), \(D = 2\) mo., \(a_S = 0\) mo., \(a_F = 2\) mo., \(d_1 = 0\) mo., \(d_2 = 0\) mo., \(p = 1\) mo., \(b = 1\) mo., \(M = 20\%\) (balanced), \(r = 10\%\) (released at 6 mo.), \(i = 5\%/\)mo. The cost slope is \(C / (D + d_2) = 250,000 / \)mo. Note that Ashley and Teicholz (1977) advocated that TVM should realistically employ two different rates; applying a contractor’s interest if the balance is negative, but using its ‘corporate rate of return’ if the balance is positive. For brevity, it is assumed that
these two rates are the same. Also, a more realistic situation of a portfolio of multiple projects is omitted and will be covered in future research. If desired, users can still calculate the breakeven point from the synthetic model (Reference 8) and then apply two different rates before and after the breakeven point. Table 1 lists results from the traditional method (with offset $\varepsilon$). Since all end balances in Table 2 are identical, the feasibility of the new synthetic method has been verified.

**Time Structure of the Model**

Time grows continuously in the model, yet individual results calculated with it are discrete. Cash outflows or inflows may occur at specific points in time, e.g. payment or interest, which are real numbers on the time axis and may be periodic events, but often coincide with integer period ends (assumed as months). Or they may grow continuously across a duration, e.g. cost (Elazouni and Metwally 2005). While Table 1 illustrates the complete arduous steps of the traditional method at each necessary point in time, Table 2 lists only the selected results in bold font to demonstrate that they match those of Table 1, while the model is able to calculate the balance at any time.

<Insert Table 1 here>

<Insert Table 2 here>

**Example for What-If Analyses**

Signal functions enable the synthetic cash flow model to perform calculations both accurately and efficiently. It incorporates various complexity-inducing factors: Bill period and bill-to-pay-delay, periodic interest if the balance is negative, withholding retainage and releasing it with the last payment, the ability to determine the required credit limit as the maximum negative balance, and full integration with the linear schedule model that also employs singularity functions.
Elazouni and Metwally (2005) provided an example with activity and cost data per Table 3. After adding values for shift and delay of each activity and the bill period, it is reanalyzed via the synthetic model. Inputting the linear schedule per Reference 3 and Figure 2a into Equations 4 to 16 yields the cash flow profile of Figure 2b. Shifting the start of activity A to 1.5 months and modifying the bill period to every 0.5 months creates a modified linear schedule of Figure 2c and cash flow profile of Figure 2d. Note that the S-shape of the cost curve for the project reflects the real-world phenomenon that few, less costly activities (e.g. mobilization) occur early in a project, then many activities simultaneously create work and generate cost in its middle, and finally few, less costly activities wrap it up (e.g. demobilization). Many factors influence the general shape of time-cost-performance, which is unique to the specific circumstances of a project. If desired, real data could be used to fine-tune singularity functions for individual activities or the entire project akin to finding a best fit polynomial in a regression analysis. This is beyond the scope of this study and may be addressed in future research. This reanalyzed example demonstrates that the synthetic model can already easily handle diverse scenarios, which fulfills Research Objective 1. Yet the model must still be expanded to incorporate unbalancing, which is a function of time.

Cumulative and Non-Cumulative Functions

Cash flow can be expressed in cumulatively or non-cumulatively; differentiating a singularity function that describes the former creates one that captures the latter (Reference 7). For example, differentiating the cumulative cost function (Equation 12) yields the non-cumulative Equation 17. Differentiating the bill function (Equation 13) gives the non-cumulative Equation 18. Taking
the difference between bill and cost functions in cumulative and non-cumulative form yields the respective markup functions. To explain further, in Example 2, change the values of Example 1 to $a_s = 1$ mo. and $a_F = 3$ mo. Entering them into the cumulative functions of Equations 12, 13, and 19 gives the profile of Figure 3; inputting them into the non-cumulative Equations 17, 18, and 19 gives Figure 4. Values in the cumulative profile are the amount up to those times, while values in the non-cumulative profile denote the instantaneously changing current rate. The total markup in Figure 3 is simply the value of the $z_{markup\_cuma}$ curve at the finish of the activity, but in Figure 4 it is the shaded area of the integral of $z_{markup\_noncum}$ along time. The merit of the plotted non-cumulative profile is that it more clearly displays if markup is unbalanced or not. Table 4 lists all cumulative and non-cumulative variables that are employed within these equations.

$$z_{\text{cost\_noncum}}(y) = z'_{\text{cost\_cuma}}(y) = \frac{C}{D + d_s} \left[ (y - a_s^0)^{\varepsilon} - (y - a_F^0)^{\varepsilon} \right]$$

$$z_{\text{bill\_noncum}}(y) = z'_{\text{bill\_cuma}}(y) = \frac{C \cdot (1 + M)}{D + d_s} \left[ (y - a_s^0)^{\varepsilon} - (y - a_F^0)^{\varepsilon} \right]$$

$$z_{\text{markup\_cuma}} = z_{\text{bill\_cuma}} - z_{\text{cost\_cuma}} \iff z_{\text{markup\_noncum}} = z'_{\text{bill\_cuma}} - z'_{\text{cost\_cuma}} = z_{\text{bill\_noncum}} - z_{\text{cost\_noncum}}$$

Unbalanced Markup Functions

The theoretical model for unbalancing the markup within the activities of a construction project is derived next. Thus multiplying the cost function with the markup factor $(1 + M)$ is discarded for a more sophisticated approach. The markup function in Equation 20 is derived to model the change over time. Three profiles are explored: Two-phase, linear, and non-linear unbalancing.
To apply these theoretical profiles in practice, a manager may analyze historic cost and schedule data to determine which profile is most suitable to their needs and past performance. The two-phase profile seeks a profit that is first high then low, or vice versa; linear sees profit as growing or shrinking throughout the project; and non-linear applies a curved behavior of profit over time.

\[ z_{\text{bill.cumulative}}(y) = z'_{\text{bill.cumulative}}(y) = \frac{C \cdot (1 + M(y))}{D + d_2} \cdot \left[ (y - a_s^*)^0 - (y - a_f^*)^0 \right] \]

**Two-Phase Markup Function**

Equation 21 provides the general two-phase non-cumulative markup function. It defines a time point \( d_3 \) somewhere between the activity start \( a_s^* \) and finish \( a_f^* \), where a high initial markup rate \( M_S \) drops to a low subsequent \( M_F \) for the remaining duration of the project. This constitutes front-loading in its simplest form; the opposite is back-loading. Total profit of the unbalanced and balanced cases must of course be equal. Based on this constraint, \( M_F \) is calculated per Equation 22. Per Federal law (23 C.F.R. § 635.114), materially unbalanced bidding is forbidden, but mathematically unbalanced bidding may be permissible. Thus the two-phase markup should be within the mathematically unbalanced range per Equation 23. Equation 24 explores the broader range for material unbalancing for completeness. Note that in Equation 24 the lowest limit of \( M_F \) is -1, because the bill must at least be zero; a negative bill would refund money to the owner.

\[ M(y) = M_S \cdot \left[ (y - a_s^*)^0 - (y - (a_s^* + d_3))^0 \right] + M_F \cdot \left[ (y - (a_s^* + d_3))^0 - (y - a_f^*)^0 \right] \]

Equation 21 is subject to the constraint \( C \cdot M_S \cdot (D + d_2) \cdot d_3 + C \cdot M_F \cdot (D + d_2) \cdot (D + d_2 - d_3) = C \cdot M_{\text{balanced}} \), where \( M_S > M_F \) for front-loading (or analogously \( M_S < M_F \) for back-loading).

\[ M_F = \frac{M_{\text{balanced}} \cdot (D + d_2) - M_S \cdot d_3}{D + d_2 - d_3} \]
Adding the constraint \( M_F \geq 0 \) \( (M_S \geq 0 \text{ if back-loading}) \) to Equation 22 so that the cost is covered provides the feasible range of \( M_S \) and \( M_F \) for mathematical unbalancing per Equation 23:

\[
M_F \geq 0 \quad \Rightarrow \quad 0 \leq M_S \leq M_{balanced} \cdot \frac{(D + d_2)}{d_3}
\]  

\text{(23)}

Adding the constraint \(-1 \leq M_F < 0 \) \( (\text{or} -1 \leq M_S < 0 \text{ if back-loading}) \) to Equation 22 so that the cost is \textit{not} covered gives the range of \( M_S \) and \( M_F \) for material unbalancing per Equation 24:

\[
-1 \leq M_F < 0 \quad \Rightarrow \quad M_{balanced} \cdot \frac{(D + d_2)}{d_3} < M_S \leq M_{balanced} \cdot \frac{(D + d_2) + (D + d_2 - d_3)}{d_3}
\]  

\text{(24)}

Adding two-phase markup parameters to Example 2 gives Example 3; let \( d_3 = 1 \text{ mo.} \), \( M_S = 30\% \), and \( M_{balanced} = 20\% \). It is mathematically unbalanced, because per Equation 22, \( M_F = (20\% \cdot 2 - 30\% \cdot 1) / (2 - 1) = 10\% \) that is larger than zero, which means the cost is covered.

Substituting the markup function per Equation 21 into the cumulative bill function per Equation 13 returns the bill function in the two-phase markup profile per Equation 25. Differentiating it generates the non-cumulative bill function. In the cumulative profile of Figure 5, the slope of the markup and bill functions change after the first month when the activity started. The unbalanced markup distribution is clearer in the non-cumulative profile of Figure 6, which shows the two phases. It is additionally verified by the fact that the bill profile always exceeds the cost profile. Equation 21 is used to calculate the maximum and minimum markup as \( M_S = 40\% \) and \( M_F = 0\% \).

For Example 4, increase the value from Example 3 to \( M_S = 60\% \). From Equation 20, \( M_F = -20\% \) that falls within the materially unbalanced markup range. Figure 7 shows the cumulative cash flow profile, where the markup function first increases and after \( d_3 \) decreases. This phenomenon is explainable with Figure 8, whose non-cumulative cash flow profile distinctly exposes the unbalanced markup and bill: First the bill profile exceeds the cost and later it is less to guarantee that the total profit remains equal to the balanced case (shaded in Figures 8 and 4).
\[ z(y)_{\text{balanced}} = \frac{C \cdot (1 + M_s)}{D + d_2} \left[ (y - a_s^*)^\dagger - (y - (a_s^* + d_1)) \right] + \frac{C \cdot (1 + M_F)}{D + d_2} \left[ (y - (a_s^* + d_1)) - (y - a_F^*)^\dagger \right] \]  

(25)

7 Linearly Distributed Markup Function

Expanding the two-phase markup function to an \( n \)-phase profile (\( n \rightarrow +\infty \)) generates a linearly distributed non-cumulative markup function per Equation 26. \( M_S \) and \( M_F \) are boundary markups at activity start and finish. For front-loading, \( M_S \) is larger than \( M_F \) (vice versa for back-loading).

\[ M(y) = M_S \cdot (y - a_s^*)^0 - M_S - M_F \cdot \left[ (y - a_s^*)^\dagger - (y - a_F^*)^\dagger \right] \]  

(26)

Equation 26 is subject to the constraint that the total profit should be constant for balanced versus unbalanced, which is written as a definite integral of the non-cumulative markup function from activity start to finish that should be equal to the balanced markup times duration:

\[ \int_{a_s^*}^{\star} \left[ M_S \cdot (y - a_s^*)^0 - (M_S - M_F) (D + d_2) \right] \left[ (y - a_s^*)^\dagger - (y - a_F^*)^\dagger \right] dy = M_{\text{balanced}} \cdot (D + d_2) \]. The area under any such non-cumulative markup function must be equal to the balanced area (Figure 4). Calculating such area \( M_S \cdot (D + d_2) - (M_S - M_F) / 2 \cdot (D + d_2) = M_{\text{balanced}} \cdot (D + d_2) \) simplifies to \( (M_S + M_F) / 2 = M_{\text{balanced}} \), where \( M_S > M_F \) for front-loading \( (M_S < M_F \) for back-loading).

Adding the constraint \( M_F \geq 0 \) \( (M_S \geq 0 \) if back-loading) to \( (M_S + M_F) / 2 = M_{\text{balanced}} \) yields Equation 27. It provides the feasible range of \( M_S \) and \( M_F \) for mathematical unbalancing:

\[ M_F \geq 0 \quad \Rightarrow \quad 0 \leq M_S \leq 2M_{\text{balanced}} \]  

(27)
Adding the constraint \(-1 \leq M_F < 0\) (-1 \leq M_S < 0 \text{ if back-loading}) to \((M_S + M_F) / 2 = M_{\text{balanced}}\) yields Equation 28. It provides the feasible range of \(M_S\) and \(M_F\) for material unbalancing:

\[-1 \leq M_F < 0 \Rightarrow 2M_{\text{balanced}} < M_S \leq 1 + 2M_{\text{balanced}}\]  \hfill (28)

For **Example 5**, use the values from **Example 1**, but for linear unbalancing. Let \(M_S = 40\%\) and \(M_F = 0\%\), which covers the cost. Substituting the markup function per Equation 26 for the obsolete bill slope term \((1 + M)\) of Equation 13 and calculating the definite integral from the activity start to the independent variable \(y\) yields the cumulative bill function per Equation 29.

\[
Z(y)_{\text{bill,cum}} = \int_{y_0}^{y} \left[ \frac{C \cdot \left[ 1 + M(t) \right]}{D + d_2} \right] dt = \frac{C}{D + d_2} \int_{y_0}^{y} \left[ 1 + M_S \cdot \left[ t - a_S^* \right]^0 - M_S - M_F \cdot \left[ (t - a_S^*)^1 - (t - a_F^*)^1 \right] \right] dt
\]

\[
= \frac{C}{D + d_2} \left[ (1 + M_S) \cdot \left[ y - a_S^* \right]^1 - M_S - M_F \cdot 2 \cdot \left[ (y - a_S^*)^2 - (y - a_F^*)^2 \right] \right] \hfill (29)
\]

Since different exponents 1 and 2 are used, Equation 29 must be equipped with terms that offset them to keep the profile stable after the activity finish per Equation 30. Similar terms are added to the non-cumulative bill function per Equation 31. Inputting \(M_S\) and \(M_F\) and plot the bill and markup functions gives Figures 9 and 10. The difference between linear (Figure 9) and two-phase markup (Figure 5) is that the former grows parabolically, because its singularity function has terms of order two in Equation 30, while the latter is stepped. Modifying \(M_S\) to its upper limit of 140\% (i.e. the maximum of the constraint \(M_S \leq 1 + 2 \cdot M_{\text{balanced}}\)) for material unbalancing and evaluating Equations 30 and 31 gives Figures 11 and 12. Figure 12 shows how the bill changes from 140\% with markup at its start to zero at its finish. As the non-cumulative markup function grows linearly, the integral for the area under the curve grows parabolically. In Figure 11, the cumulative markup function thus first increases then decreases. Its apex occurs when the non-cumulative markup function switches its sign from positive to negative. Cost is not covered in the materially unbalanced scenario, as the area later decreases due to the negative markup.
\[ z(y)_{\text{hill\_cumu}} = \frac{C}{D+d_z} \left[ (1+M_s) \left( \langle y-a^*_s \rangle^0 - \langle y-a^*_F \rangle^0 \right) - \frac{M_s-M_F}{2(D+d_z)} \left( \langle y-a^*_s \rangle^2 - \langle y-a^*_F \rangle^2 - (D+d_z) \langle y-a^*_F \rangle^0 \right) \right] \]

\[ z(y)_{\text{hill\_noncumu}} = \frac{C}{D+d_z} \left[ (1+M_s) \left( \langle y-a^*_s \rangle^0 - \langle y-a^*_F \rangle^0 \right) - \frac{M_s-M_F}{D+d_z} \right] \left( \langle y-a^*_s \rangle^1 - \langle y-a^*_F \rangle^1 \right) - (D+d_z) \left( \langle y-a^*_F \rangle^0 \right) \]  

Equation 30

Equation 31

Non-Linear Distributed Markup Function

The two-phase and linear markups per Equations 21 and 26 only had order 0 or 1. However, it is theoretically possible for unbalanced markups to grow non-linearly. It bears resemblance to the depreciation methods that equipment owners may use, which may be straight-line or non-linear to accelerate the loss of book value (Reference 1). Equation 32 is a second-order non-linear non-cumulative markup function. Higher exponents can be modeled, but are omitted for brevity.

\[ M(y) = A \cdot \langle y-a^*_s \rangle^0 + B \cdot \left( \langle y-a^*_s \rangle^1 - \langle y-a^*_F \rangle^1 \right) + C \cdot \left( \langle y-a^*_s \rangle^2 - \langle y-a^*_F \rangle^2 \right) \]  

Equation 32

Note that for front-loading, the markup function must go through the points \{ a^*_s, M_s \} and \{ a^*_F, M_F \}, where \( M_S \) is larger than \( M_F \) (smaller for back-loading). Multiple coefficients \( A, B, \) and \( C \) exist, which allow an unlimited number of shapes. Here only one possible solution is shown, which sets the first order term to zero for illustration. The aforementioned constraint applies that the integral of the non-linear and balanced cases are equal. Evaluating Equation 32
for \( A = M_S, B = 0, \) and \( C = \frac{1}{(D + d_z)^2} \cdot (M_F - M_S) \) provides an example of a user-customized markup function \( M(y) = M_S \langle y - a_s^* \rangle - 1/(D + d_z)^2 \cdot (M_S - M_F) \cdot \left[ \langle y - a_s^* \rangle^2 - \langle y - a_F^* \rangle^2 \right] \). It can be integrated to
\[
\int_{a_s}^{y} \left( M_S \langle y - a_s^* \rangle - 1/(D + d_z)^2 \cdot (M_S - M_F) \cdot \left[ \langle y - a_s^* \rangle^2 - \langle y - a_F^* \rangle^2 \right] \right) dy = M_{\text{balanced}} \cdot (D + d_z) ,
\]
which simplifies to \( 2/3 \cdot M_S + 1/3 \cdot M_F = M_{\text{balanced}} \).

Adding the constraint \( M_F \geq 0 \) \((M_S \geq 0 \text{ if back-loading})\) to \( 2/3 \cdot M_S + 1/3 \cdot M_F = M_{\text{balanced}} \) yields Equation 33. It provides the feasible range of \( M_F \) and \( M_S \) for mathematical unbalancing:

\[
M_F \geq 0 \implies 0 \leq M_S \leq \frac{3}{2} M_{\text{balanced}} \quad \text{(33)}
\]

Adding the constraints \(-1 \leq M_F < 0 \) \((-1 \leq M_S < 0 \text{ if back-loading})\) to \( 2/3 \cdot M_S + 1/3 \cdot M_F = M_{\text{balanced}} \) yields Equation 34. It provides the feasible range of \( M_F \) and \( M_S \) for material unbalancing:

\[
-1 \leq M_F < 0 \implies \frac{3}{2} M_{\text{balanced}} < M_S \leq \frac{3M_{\text{balanced}} + 1}{2} \quad \text{(34)}
\]

For Example 6, use the values from Example 1, but calculate non-linear unbalancing.

For mathematical unbalancing, let \( M_S = 30\% \) and \( M_F = 0\% \), which covers the cost. Applying the definite integral from the activity start to the independent variable \( y \) yields the cumulative bill function per Equation 35. With the stop terms it becomes Equation 36; differentiating it generates the non-cumulative bill function per Equation 37, both of which are shown in Figures 13 and 14. The ranges of markup values for material unbalancing per Equation 34 are \(-1 \leq M_F < 0 \) and \( 30\% < M_S \leq 80\% \). Entering the boundary values of \( M_F = -1 \) per the constraint \( 2/3 \cdot M_S + 1/3 \cdot M_F = M_{\text{balanced}} \) and \( M_S = 80\% \) into Equations 36 and 37 gives Figures 15 and 16.

\[
Z(y)_{\text{cumulative}} = \int_{a_s}^{y} C \cdot \left[ 1 + M(t) \right] dt = \frac{C}{D + d_z} \cdot \int_{a_s}^{y} \left[ 1 + M_S \cdot \langle t - a_s^* \rangle^6 \right] - \frac{1}{(D + d_z)^2} \cdot (M_S - M_F) \cdot \left[ \langle t - a_s^* \rangle^2 - \langle t - a_F^* \rangle^2 \right] dt
\]
\[
= \frac{C}{D + d_z} \cdot \left[ 1 + M_S \right] \cdot \left[ \langle y - a_s^* \rangle \right] - \frac{M_S - M_F}{3(D + d_z)^2} \cdot \left[ \langle y - a_s^* \rangle^3 - \langle y - a_F^* \rangle^3 \right] \quad \text{(35)}
\]
Overall, the unbalanced markup function can accurately and efficiently model different profiles, including two-phase, linear, and even non-linear, which fulfills Research Objective 2. Markup functions can be applied to any activity duration that exceeds one month (unbalancing would not change a single total bill). The synthetic model is updated in the following section.

Synthetic Cash Flow Model for Unbalanced Markups

Replace the fixed markup parameter $M$ by the markup function $M(y)$ in the synthetic cash flow model builds the new cash flow model for unbalanced markup scenarios. A difference between the unbalanced and balanced model is how progress pay is modeled. Equation 6 multiplies a pay signal function with the bill slope to calculate each pay, because in a balanced scenario the time and cost are proportional. However, the bill grows non-linearly in an unbalanced scenario, so that such proportionality exists no longer. Accordingly using Equation 6 in unbalanced scenarios
would generate an incorrect pay. Instead a different approach is used, subtracting the value of the cumulative pay function at a later time from the value of the same function at an earlier time.

**Synthetic Model for Two-Phase Markup**

Equation 38 is the cumulative pay function. Its rounddown operator creates a stepped profile of incremental bills and $b$ is subtracted to shift the bill to become pay. The rest of the synthetic model remains the same as before. For **Example 7**, use the values from **Example 1**, but applying the two-phase unbalancing. For mathematical unbalancing, let $d_3 = 1$ mo., $M_S = 30\%$, and $M_F = (20\% \cdot 2 - 30\% \cdot 1) / (2 - 1) = 10\%$ (calculated per Equation 21). Table 5 shows balances per the synthetic model, which applies the distributive law to shorten its calculation without sacrificing accuracy. The much longer calculation with the chronological method is omitted here for brevity.

$$
\begin{align*}
\text{Equation 38:} & \quad z(y)_{\text{pay-cum}} = \frac{C \cdot [1 + M_S(y)]}{D + d_z} \cdot p \cdot \left[ \left( \frac{y - b}{p} \right) - \left( \frac{a_s^*}{p} \right) \right] \\
& + \frac{C \cdot [1 + M_F(y)]}{D + d_z} \cdot p \cdot \left[ \left( \frac{y - b}{p} \right) - \left( \frac{a_s^* + d_3}{p} \right) \right]
\end{align*}
$$

<Insert Table 5 here>

**Synthetic Model for Linear Markup**

Similar to before, updating the pay function for the linear markup profile per Equation 39 uses a rounddown operator and shift. For **Example 8**, use the values from **Example 1**, but applying linear unbalancing. For mathematical unbalancing, let $M_S = 30\%$ and $M_F = 20\% \cdot 2 - 30\% = 10\%$ (calculated per Equation 25). Table 6 lists the detailed cash flow calculations.
Synthetic Model for Non-linear Markup

The cumulative pay function is given per Equation 40 by using the same treatment as the previous two markup cases. Again, a tabular calculation is performed per Table 7. For Example 9, use the values from Example 1, but applying nonlinear unbalancing. For mathematical unbalancing, let \( M_S = 30\% \) and \( M_F = 20\% \cdot 3 - 30\% \cdot 2 = 0\% \) (calculated per Equation 31).

Comparison between Different Unbalanced Bidding Scenarios

Different types of unbalanced markups only influence the amount of each pay per Tables 1, 5, 6, and 7. Plotting the cash flow example by Elazouni and Metwally (2005) using different markup functions returns the profiles of Figures 17, respectively. Non-cumulative bill functions for each scenario are also shown to distinguish markup types. For comparison, the PV of the balance at
any time are also included per Equation 41. Table 8 lists the PV of all balances at the project finish for the different scenarios. Comparing scenarios shows that the two-phase markup has the maximum PV for \( d_3 = 50\% \) of duration for the two-phase markup and \( M_S = 40\% \). Comparing bill-to-pay delays shows that the smaller the bill-to-pay-delay, the larger the PV. Differences in PV are large, because a rather high interest rate of 5\% / mo. was assumed in the original example.

\[
z_{PV\_balance}(y) = z_{balance}(y) \cdot (1 + i)^y
\]  
(41)

<Insert Figure 17 here>

<Insert Table 8 here>

In sum, diverse unbalanced bidding markup functions are integrated into the synthetic cash flow model accordingly and verified by tabular calculation examples. The mathematical formularized cash flow models allow numerous ‘what-if’ analyses by manipulating different parameter inputs of the model. These results show the flexibility of the synthetic cash flow model for both balanced and unbalanced bidding scenarios, which fulfills Research Objective 3.

Conclusions and Contributions to the Body of Knowledge

Different from previous unbalanced bidding research that locate different unit price on work items, the focus of this study are unbalanced bidding situations at the detailed level of activities, which according to the possibility that different cost items may distribute unevenly along the activity duration, the cash flow profiles, e.g. bill, may possess a nonlinear shape. Three Research Objectives have been fulfilled. First, based on the previous research (Reference 8), the synthetic cash flow model has been expanded with retainage subtracting and releasing conditions, which anchored as the baseline for the following unbalanced bidding scenarios. Second, markup
function has been defined as a function of time, and the cumulative and non-cumulative markup functions and profiles have been also defined and plotted to assist the unbalanced bidding research. Three unbalanced bidding markup types; two-phase, linear, and nonlinear markup functions have been defined for both mathematically and materially unbalanced bidding cases. Third, the synthetic cash flow model has been updated with distinct unbalanced bidding markup functions. The present value of the cash flow balances at any time can also be calculated with the model. By changing parameter values, a sensitive analysis can identify appropriate unbalancing.

**Contributions to the Body of Knowledge** in construction management include that a new mathematical way has been derived to model unbalanced bidding within activities. Markups of three possible profiles have been modeled: Two-phase, linear, and non-linear growth of the markup, the latter so that higher order shapes can also be expressed, if needed. They now allow a virtually unlimited customization. The synthetic cash flow model connects unbalanced bidding directly with an underlying project schedule. Mathematically and materially unbalanced bids can be calculated accurately and efficiently. Singularity functions have thus been upgraded to fit more complicated situations in both depth and breadth of the construction management domain.

**Recommendations for Future Research**

Green (1989, p. 54) noted that ideally “labour, plant [equipment], materials and subcontractor elements would be evaluated in order to produce the net cost estimate. Allowances for overheads, profit and risk would be made by management at the tender adjudication stage.” Future research could thus explore strategies how and when to fine-tune the markup for “operation-related cost items (such as labor and equipment cost)” (Wang 2004, p. 458) and how such modifications
interact with the risk for the owner and contractor. The new cash flow model could be detailed so
that the markup would depend on both cost type and time, which would add even more realism.

Featuring the extensibility that is inherent to singularity functions, which can model any
range-based functions, the new model for unbalanced bidding can be integrated easily with cash
flows (Reference 3), linear scheduling (Reference 2), and resource use of a construction project,
which opens a gate to exploring the unbalanced bidding type of individual rate loading. It could
use a 3D coordinate system with time, cost, and work quantity axes (Reference 5). The
difference between an owner’s estimate of relative duration and unit cost and a bidder’s estimate
of quantity and duration with individual rate loading of cost could thus be calculated. Performing
such research could further improve the understanding of unbalanced bidding mechanisms.

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[Reference 1 hidden to anonymous peer-review]

[Reference 2 hidden to anonymous peer-review]

[Reference 3 hidden to anonymous peer-review]

[Reference 4 hidden to anonymous peer-review]


Notation

Symbols

1. \( a \) = activation cutoff in singularity function;
2. \( b \) = bill-to-pay delay;
3. \( C \) = total cost of an activity;
4. \( d \) = variable affects start or finish of an activity (shift or delay);
5. \( D \) = scheduled duration of activity;
6. \( i \) = financing interest in percent of balance per period;
7. \( M \) = balanced distributed markup for profit;
8. \( n \) = shape exponent in singularity function;
9. \( p \) = bill period;
10. \( r \) = retainage;
11. \( s \) = scaling factor in singularity function;
12. \( y \) = independent variable along horizontal axis, here time;
13. \( z \) = dependent variable along vertical axis, here money;
14. \( \varepsilon \) = infinitesimal duration offset;
15. \( \langle \rangle \) = brackets of singularity functions;
16. \( \lfloor \rfloor \) = floor operator rounding downward to integer;
17. \( \lceil \rceil \) = ceiling operator rounding upward to integer.

Subscripts

1. \( \text{activity} \) = activity;
2. \( \text{bill\_cumu} \) = cumulative bill function;
1 bill_noncumu = noncumulative bill function;
2 balance = balance function with considering time value of money;
3 balanced = balanced bidding;
4 cost = cost function without considering time value of money;
5 cost_cumu = cumulative cost function;
6 cost_int = future value of cost with compound interest
7 cost_noncumu = noncumulative cost function;
8 each_pay = pay amount at each pay time;
9 F = finish of activity;
10 FV_pay = future value of accumulated amount of pay at a time;
11 int_at_signal = charged interest on cost at each signal time;
12 int_signal = signal at each charging interest time;
13 last_pay = last pay function;
14 markup_cumu = cumulative markup function;
15 markup_noncumu = noncumulative markup function;
16 pay = pay function;
17 pay_cumu = cumulative pay function;
18 pay_with_ret = pay with retainage;
19 pay_signal = signal at each pay time;
20 project = project;
21 PV_balance = present value of balance;
22 ret = retainage;
23 S = start of an activity;
1  step_cost = step function of cost which return the cumulated cost only at the end of
2  each period;
3  1 = index for time shift or markup for profit at activity start;
4  2 = index for time delay or markup for profit at activity finish;

5  **Superscripts**
6  * = variable that includes shifts and/or delays.
Figure 1: Structure of Modeling Approach

**Existing Scheduling Model**

Components:
- Activities, buffers
- Duration, shift, delay
- Calculate starts, finishes
- Calculate criticality, float

**Periodicity**

Signal Function:
- Interest signal
- Payment signal
- Retainage signal

**Synthetic Cash Flow Model**

Components:
- Cost
- Bill
- Financing interest

**Integrated Model for Cash Flow and Schedule**

- Progress payment
- Retainage withholding
- Balance at any time
- Present and future value

**Research Objective 1**

Synthetic Cash Flow Model for Comparison

**Research Objective 2**

Material Mathematical

**Research Objective 3**

Synthetic Cash Flow Model for Unbalanced Bidding Scenarios

Balanced Bidding $M(t) = \text{Constant}$

Unbalanced Bidding $M(t) \neq \text{Constant}$

Markup as Function of Time $M(t)$

Markup Profiles:
- Two-phase $M(t)$
- Linear $M(t)$
- Non-linear $M(t)$

Singularity Functions as Common Enabling Factor
- Progress payment
- Retainage withholding
- Balance at any time
- Present and future value
Figure 2: Linear Schedules and Cash Flow Profiles for ‘What-If’ Analysis (applied data from Elazouni and Metwally 2005)
Figure 3: Cumulative Profiles of Balanced Markup
Figure 4: Non-Cumulative Profiles of Balanced Markup
Figure 5: Cumulative Profiles of Two-Phase Mathematical Unbalancing
Figure 6: Non-Cumulative Profiles of Two-Phase Mathematical Unbalancing
Figure 7: Cumulative Profiles of Two-Phase Material Unbalancing
Figure 8: Non-Cumulative Profiles of Two-Phase Material Unbalancing
Figure 9: Cumulative Profiles of Linear Mathematical Unbalancing
Figure 10: Non-Cumulative Profiles of Linear Mathematical Unbalancing
Figure 11: Cumulative Profiles of Linear Material Unbalancing
Figure 12: Non-Cumulative Profiles of Linear Material Unbalancing
Figure 13: Cumulative Profiles of Non-Linear Mathematical Unbalancing
Figure 14: Non-Cumulative Profiles of Non-Linear Mathematical Unbalancing
Figure 15: Cumulative Profiles of Non-Linear Material Unbalancing
Figure 16: Non-Cumulative Profiles of Non-Linear Material Unbalancing
a: Balanced Markup,
  \( M_{\text{balance}} = 20\% \)

b: Two-Phase Markup,
  \( d_3 = 50\% \) of duration, \( M_S = 40\% \)

c: Linear Markup, \( M_S = 40\% \)

d: Non-Linear Markup, \( M_S = 40\% \)

Figure 17: Cash Flow Profiles for Different Unbalanced Bidding Types
(applied data from Elazouni and Metwally 2005)
Table 1: Balance with Interest Using Infinitesimal Offset $\varepsilon$

<table>
<thead>
<tr>
<th>$y$ [months]</th>
<th>Cost [$1,000$]</th>
<th>Interest [$1,000$]</th>
<th>Payment [$1,000$]</th>
<th>Balance [$1,000$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-250</td>
<td>0</td>
<td>0</td>
<td>-250.000</td>
</tr>
<tr>
<td>$1 + \varepsilon$</td>
<td>0</td>
<td>$250 \cdot i / \ln(1 + i) - 250 = 6.1992$</td>
<td>0</td>
<td>-256.199</td>
</tr>
<tr>
<td>$1 + 2\varepsilon$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-256.199</td>
</tr>
<tr>
<td>2</td>
<td>-250</td>
<td>0</td>
<td>0</td>
<td>-506.199</td>
</tr>
<tr>
<td>$2 + \varepsilon$</td>
<td>0</td>
<td>$256.199 \cdot i + 250 \cdot i / \ln(1 + i) - 250 = 12.8100 + 6.1992 = 19.0092$</td>
<td>0</td>
<td>-525.208</td>
</tr>
<tr>
<td>$2 + 2\varepsilon$</td>
<td>0</td>
<td>0</td>
<td>250 \cdot 1.2 \cdot 0.9 = 270</td>
<td>-552.208</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-255.208</td>
</tr>
<tr>
<td>$3 + \varepsilon$</td>
<td>0</td>
<td>$255.208 \cdot i = 12.7604$</td>
<td>0</td>
<td>-267.968</td>
</tr>
<tr>
<td>$3 + 2\varepsilon$</td>
<td>0</td>
<td>0</td>
<td>250 \cdot 1.2 \cdot 0.9 = 270</td>
<td>2.032</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.032</td>
</tr>
<tr>
<td>$4 + \varepsilon$</td>
<td>0</td>
<td>$2.032 \cdot i = 0.1016$</td>
<td>0</td>
<td>2.134</td>
</tr>
<tr>
<td>$4 + 2\varepsilon$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.134</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.134</td>
</tr>
<tr>
<td>$5 + \varepsilon$</td>
<td>0</td>
<td>$2.134 \cdot i = 0.1067$</td>
<td>0</td>
<td>2.241</td>
</tr>
<tr>
<td>$5 + 2\varepsilon$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.241</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.241</td>
</tr>
<tr>
<td>$6 + \varepsilon$</td>
<td>0</td>
<td>$2.241 \cdot i = 0.1121$</td>
<td>0</td>
<td>2.353</td>
</tr>
<tr>
<td>$6 + 2\varepsilon$</td>
<td>0</td>
<td>0</td>
<td>500 \cdot 1.2 \cdot 0.1 = 60</td>
<td>62.353</td>
</tr>
</tbody>
</table>

*Note: Compare bold values with Table 2*
Table 2: Balance with Interest Using Future Values of Payment and Cost

<table>
<thead>
<tr>
<th>$y$ [months]</th>
<th>Future Value of Payment (Step 1) [$1,000]</th>
<th>Future Value of Cost (Step 2) [$1,000]</th>
<th>Balance [$1,000]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00</td>
<td>-250 $\cdot \frac{[1 + (1 + i)^1] - 1}{\ln(1 + i)}$ = -256.199</td>
<td>-256.199</td>
</tr>
<tr>
<td>2</td>
<td>250 $\cdot 1.2 \cdot 0.9 = 270$</td>
<td>-250 $\cdot \frac{[1 + (1 + i)^2] - 1}{\ln(1 + i)}$ = -525.208</td>
<td>-525.208</td>
</tr>
<tr>
<td>3</td>
<td>$270 \cdot (1 + i)^1 + 270$ = 553.5</td>
<td>-525.208 $\cdot (1 + i)^1 = -551.468$</td>
<td>2.032</td>
</tr>
<tr>
<td>4</td>
<td>$553.5 \cdot (1 + i)^1 = 581.175$</td>
<td>-551.468 $\cdot (1 + i)^1 = -579.041$</td>
<td>2.134</td>
</tr>
<tr>
<td>5</td>
<td>$581.175 \cdot (1 + i)^1 = 610.234$</td>
<td>-579.041 $\cdot (1 + i)^1 = -607.993$</td>
<td>2.241</td>
</tr>
<tr>
<td>6</td>
<td>$610.234 \cdot (1 + i)^1 + 60 = 700.746$</td>
<td>-607.993 $\cdot (1 + i)^1 = -638.393$</td>
<td>62.353</td>
</tr>
</tbody>
</table>

Note: Compare bold values with Table 1
Table 3: Activity List and Direct Cost for Initial Configuration
(adapted from Elazouni and Metwally 2005, Reference 3)

<table>
<thead>
<tr>
<th>Name</th>
<th>$D$ [months]</th>
<th>$a_S$ [month]</th>
<th>$a_F$ [month]</th>
<th>Predecessor [-]</th>
<th>Successor [-]</th>
<th>Unit Cost [$/month]</th>
<th>$C$ [$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>Start</td>
<td>C, D, F</td>
<td>100,000</td>
<td>100,000</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>Start</td>
<td>C, E</td>
<td>105,000</td>
<td>210,000</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>A, B</td>
<td>Finish</td>
<td>110,000</td>
<td>110,000</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>A</td>
<td>Finish</td>
<td>105,000</td>
<td>105,000</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>B</td>
<td>Finish</td>
<td>115,000</td>
<td>345,000</td>
</tr>
<tr>
<td>F</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>A</td>
<td>Finish</td>
<td>105,000</td>
<td>210,000</td>
</tr>
</tbody>
</table>

Note: $d_1 = d_2 = 0; p = 1$ mo.; $b = 1$ mo.; $i = 0.8\%/$mo.; and $M = 20\%$. 
<table>
<thead>
<tr>
<th>Cumulative Variables</th>
<th>Non-Cumulative Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_{\text{balance}}$</td>
<td>$z_{\text{ret}}$</td>
</tr>
<tr>
<td>$z_{\text{bill cumu}}$</td>
<td>$z_{\text{bill noncumu}}$</td>
</tr>
<tr>
<td>$z_{\text{cost cumu}}$</td>
<td>$z_{\text{cost noncumu}}$</td>
</tr>
<tr>
<td>$z_{\text{cost int}}$</td>
<td>$z_{\text{each pay}}$</td>
</tr>
<tr>
<td>$z_{\text{FV pay}}$</td>
<td>$z_{\text{last pay}}$</td>
</tr>
<tr>
<td>$z_{\text{int at signal}}$</td>
<td>$z_{\text{int signal}}$</td>
</tr>
<tr>
<td>$z_{\text{markup cumu}}$</td>
<td>$z_{\text{markup noncumu}}$</td>
</tr>
<tr>
<td>$z_{\text{PV balance}}$</td>
<td>$z_{\text{pay signal}}$</td>
</tr>
</tbody>
</table>
Table 5: Cash Flow Calculation for Two-phase Markup Profile

<table>
<thead>
<tr>
<th>$y$ [months]</th>
<th>Future Value of Payment (Step 1) [$1,000]</th>
<th>Future Value of Cost (Step 2) [$1,000]</th>
<th>Balance [$1,000]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00</td>
<td>-250 \cdot \frac{[(1 + i)^1 - 1]}{\ln(1 + i)} = -256.199</td>
<td>-256.199</td>
</tr>
<tr>
<td>2</td>
<td>250 \cdot 1.3 \cdot 0.9 = 292.5</td>
<td>-250 \cdot \frac{[(1 + i)^2 - 1]}{\ln(1 + i)} = -232.708</td>
<td>-232.708</td>
</tr>
<tr>
<td>3</td>
<td>292.5 \cdot (1 + i)^1 + 250 \cdot 1.1 \cdot 0.9 = 554.625</td>
<td>-525.208 \cdot (1 + i)^1 = -551.468</td>
<td>3.157</td>
</tr>
<tr>
<td>4</td>
<td>554.625 \cdot (1 + i)^1 = 582.356</td>
<td>-551.468 \cdot (1 + i)^1 = -579.041</td>
<td>3.315</td>
</tr>
<tr>
<td>5</td>
<td>582.356 \cdot (1 + i)^1 = 611.474</td>
<td>-579.041 \cdot (1 + i)^1 = -607.993</td>
<td>3.481</td>
</tr>
<tr>
<td>6</td>
<td>611.474 \cdot (1 + i)^1 + 60 = 702.048</td>
<td>-607.993 \cdot (1 + i)^1 = -638.393</td>
<td>63.655</td>
</tr>
</tbody>
</table>
Table 6: Cash Flow Calculation for Linear Markup Profile

<table>
<thead>
<tr>
<th>y [months]</th>
<th>Future Value of Payment (Step 1) [$1,000]</th>
<th>Future Value of Cost (Step 2) [$1,000]</th>
<th>Balance [$1,000]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00</td>
<td>-250 \cdot [(1 + i)^1 - 1] / \ln(1 + i) = -256.199</td>
<td>-256.199</td>
</tr>
<tr>
<td>2</td>
<td>\frac{500}{2} \cdot \left[ 1 \cdot (1+30%) - 1^2 \cdot (30% - 10%) / (2 \cdot 2) \right] \cdot 0.9 = 281.25 (use Equation 39 \cdot r)</td>
<td>-250 \cdot [(1 + i)^2 - 1] / \ln(1 + i) = -525.208</td>
<td>-243.958</td>
</tr>
<tr>
<td>3</td>
<td>281.25 \cdot (1 + i)^1 + (500 \cdot 1.2 \cdot 0.9 - 281.25) = 554.062</td>
<td>-525.208 \cdot (1 + i)^1 = -551.468</td>
<td>2.594</td>
</tr>
<tr>
<td>4</td>
<td>554.062 \cdot (1 + i)^1 = 581.765</td>
<td>-551.468 \cdot (1 + i)^1 = -579.041</td>
<td>2.724</td>
</tr>
<tr>
<td>5</td>
<td>581.765 \cdot (1 + i)^1 = 610.853</td>
<td>-579.041 \cdot (1 + i)^1 = -607.993</td>
<td>2.860</td>
</tr>
</tbody>
</table>
### Table 7: Cash Flow Calculation for Nonlinear Markup Profile

<table>
<thead>
<tr>
<th>$y$ [months]</th>
<th>Future Value of Payment (Step 1) [$1,000]</th>
<th>Future Value of Cost (Step 2) [$1,000]</th>
<th>Balance [$1,000]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00</td>
<td>-$250 · [(1 + $i^1) - 1] / ln(1 + $i)$</td>
<td>-256.199</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$250 · [1 + (30%) - 1^1] / (3 · 2^2)$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>500 / 2 · [1 · (1+30%) - 1^2 · (30% - 0%) / (3 · 2^2)] · 0.9 = 286.875 (use Equation 40 · $r$)</td>
<td>-$250 · [(1 + $i^2) - 1] / ln(1 + $i)$ = -525.208</td>
<td>-238.333</td>
</tr>
<tr>
<td>3</td>
<td>286.875 · (1 + $i^1$) + (500 · 1.2 · 0.9 - 286.875) = 554.343</td>
<td>-$525.208 · (1 + $i^1$) = -551.468</td>
<td>2.875</td>
</tr>
<tr>
<td>4</td>
<td>554.343 · (1 + $i^1$) = 582.060</td>
<td>-$551.468 · (1 + $i^1$) = -579.041</td>
<td>3.019</td>
</tr>
<tr>
<td>5</td>
<td>582.06 · (1 + $i^1$) = 611.163</td>
<td>-$579.041 · (1 + $i^1$) = -607.993</td>
<td>3.170</td>
</tr>
<tr>
<td>6</td>
<td>611.163 · (1 + $i^1$) + 60 = 701.721</td>
<td>-$607.993 · (1 + $i^1$) = -638.393</td>
<td>63.328</td>
</tr>
</tbody>
</table>
Table 8: Present Value of Balances for Different Bill-to-Pay Delay and Markup Profiles

<table>
<thead>
<tr>
<th>Bill-to-Pay Delay</th>
<th>Present Value of Balance, ( M_s = 40% ) [1,000]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Balanced ( d_3 = 50% ) of duration</td>
</tr>
<tr>
<td>( b = 1 ) mo.</td>
<td>100.298</td>
</tr>
<tr>
<td>( b = 2 ) mo.</td>
<td>49.126</td>
</tr>
<tr>
<td>( b = 3 ) mo.</td>
<td>0.391</td>
</tr>
</tbody>
</table>
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