Abstract

Construction managers plan and control quantitative measures for successful projects. Current techniques for time, budget, and resources are compartmentalized, but should be integrated into a cohesive model. The methodology adapts singularity functions from structural engineering. They activate a dependent variable over a range of an independent variable. Yet their analytical capability had ignored interfaces between measures. Pairwise interactions link quantitative elements and enable a customizable approach that facilitates multi-objective optimization. This research contributes in that models of time, cost, and resources are aligned; interactions are formalized via the common variable time; and the possibility of higher order models is discussed.

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Keywords: Cash flows; integrated project management; quantitative performance measures; scheduling; singularity functions

1. Introduction

Construction project management is a complex system, as it is driven by multiple objectives; “[t]hese objectives and their relative importance vary from one project to another, and they often include minimizing construction time and cost while maximizing safety, quality, and sustainability” [1, p. 17]. The term ‘objective’ can also be used interchangeably to the term dimension, as traditional project management is commonly defined as a three-dimensional (3D) cost-schedule-technical system [2]. To handle the characteristics of such a complex system, previous studies divided construction projects into various subsystems. They typically focused on one compartmentalized subsystem as their research purpose, while simplifying or even omitting interfaces to other subsystems. For example, Gantt bar charts are a graphical scheduling method that is oriented exclusively toward the time dimension, ignoring others such as work quantity, cost, and resources. The well-known Critical Path Method (CPM) was developed for the re-
requirement that “[t]he plan should point directly to the difficult and significant activities – the problems of achieving the objective” [3, p. 160]. CPM may be described as a one-and-a-half dimensional method, because it primarily applies an algorithm to schedule time, while a separate later time-cost tradeoff analysis could consider direct cost, but not indirect cost [3]. More recently, additional dimensions (often three) are brought into multi-objective models. They were solved “using a variety of methods, including linear programming, integer programming, dynamic programming, and genetic algorithms” [4, p. 477]. But CPM still dominates as their limiting foundation to the detriment of other dimensions. Yet construction projects are integrated systems that unfold in a complex interplay subject to a plethora of factors, which requires using a more integrated model to generate realistic analyses and efficient optimization. A need exists to explore novel approaches toward such an integrated systems view of project management.

2. Literature Review

Traditional research “selected [the] most important objective while either neglecting the less important competing objectives or imposing them as known constraints in the optimization formulation” [5, p. 1411]. Such studies on single objectives usually explore a detailed subsystem, e.g. scheduling [6] or cash flows [7]. But construction projects are complex systems and thus it is deplorable that there exists “a plethora of “control” techniques that cannot provide any insight into the interactions among the many components of a construction engineering project” [8, p. 494]. To manage multi-objective problems, studies seek to create “tools and strategies that can simultaneously improve project performance in multiple dimensions” for integrated systems [9, p. 30]. Yet “[m]ultiobjective optimization formulations have clear theoretical advantages but increase the complexity of the mathematical formulation” [5, p. 1411]. In most cases, researchers make approximations to simplify problems. Ammar [10, p. 67] for a time-cost tradeoff optimization explained that a “[d]iscount factor in the exponential form…, is too complicated to be handled in a mathematical optimization model… Instead, a simplified form… will be used.” Such approach is common in multi-objective research for simplicity, manageability, and brevity of a model and its description, but undesirable.

Furthermore, such studies typically follow very similar steps: First, selecting objectives from project performance parameters to be minimized or maximized, e.g. time, cost, or resources. Second, using a multi-objective optimization algorithm. However, an important intermediate step is often short-changed, that of creating a model whose nature is ideally suited to its challenge. It logically occurs between establishing the objectives and performing an optimization and is crucial for an efficient and reliable optimization. Models must be versatile yet accurate to the maximum extent that input data allow, without imposing extraneous restrictions from modeling assumptions. Prior studies have focused extensively on optimization algorithms [11], but appear to overlook this modeling challenge.

2.1. Research Need and Objectives

Abridged objective functions to minimize or maximize dependent variables radically simplify reality: Duration is determined by factors such as productivity, crew size, resource availability, shifts, lead/lag durations, buffers, etc.; multi-objective studies omit most such details. Cash flows must consider direct and indirect cost, bill-to-pay delay, prompt payment discount, credit limit, interest, time value of money, etc. Objective functions typically simplify these details, focusing instead on algorithms to identify or compare solutions. One may argue that it is overly complicated to maintain the same level of detail as local subsystems when moving to modeling a global integrated system. But per Ockham’s razor, whose paradigm advocated ‘as simple as possible, as complicated as necessary’ for models, one should not reduce realism if a system becomes more complex, which limits the validity of its output and may mislead decision-makers. This research raises the fundamental question of how to bridge local and global views, while remaining efficient and accurate as determined by the quality of available data, not model assumptions. Singularity functions, defined in the following, offer the unique features of detailability (can reflect any desired level of detail within their mathematical expressions), extensibility (can incorporate any number of interacting dimensional variables), and convertibility (can extract pairwise performance parameters to examine their relationship).

Three sequential Research Objectives will be addressed by this research, which together contribute to its overall goal of ultimately gaining a single comprehensive yet customizable approach to construction project management:

- Exploring interactions and conversions among singularity functions for merging subsystems into a global model;
• Aligning subsystem schedule, cash flow, and resource models in cumulative and non-cumulative expressions;
• Visualizing the integrated 3D project model and assessing its potential contribution as a novel management tool.

3. Singularity Functions

Singularity functions were initially used in structural engineering to analyze the problem of beams that are loaded with diverse types of loads, which are located at discrete or distributed locations on said beams. The basic term of Equation 1 was historically known as the Föppl symbol [12] or Macaulay brackets [13]. This operator performs a case distinction for the given cutoff value \( a \) of the independent variable \( x \). The functions yields zero if \( x \) is smaller than \( a \), while evaluating the pointed brackets as round ones if \( x \) is equal to or larger than \( a \). The strength \( s \) amplifies the value of the function, while the exponent \( n \) modifies the behavior to linear or nonlinear growth of the function.

The virtue of singularity functions is that they can easily model influences that are located at or distributed along a dimension (e.g. time), which fulfills requirements of some key applications in the construction management field. For example, in linear schedules, activity progress is represented as a curve with start and finish date, whose slope is the productivity. Moreover, for cash flows, a cost profile also has a start and finish; its slope is the rate at which the cost grows. Furthermore, for resources utilization, a profile can be derived analogously. Of course, multiple effects of the same type can be expressed jointly, which generates a staggered profile over its respective time period. All of such phenomena can be modeled with the basic term of singularity functions by inserting the respective appropriate independent and dependent variables per Table 1. This feature enables the conversion per Research Objective 1: Introducing a new variable, e.g. a cost factor for work units into a linear schedule, transforms the model as desired.

\[
s \cdot (x-a)^n = \begin{cases} 
0 & \text{for } x < a \\
\frac{s \cdot (x-a)^n}{a} & \text{for } x \geq a 
\end{cases}
\]  

(1)

Table 1. Variables for singularity functions.

<table>
<thead>
<tr>
<th>Type of problem</th>
<th>Independent</th>
<th>Dependent</th>
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<tr>
<td>Linear schedule</td>
<td>( x = ) work unit</td>
<td>( y = ) time unit</td>
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<tr>
<td>Cash flows</td>
<td>( y = ) time unit</td>
<td>( z = ) cost unit</td>
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<tr>
<td>Resource utilization</td>
<td>( y = ) time unit</td>
<td>( r = ) resource unit</td>
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3.1. Linear Schedule model (cumulative and non-cumulative forms)

Most construction projects, whether pipelines or multi-floor buildings, contain many repeating activities [6]. CPM can only guarantee that sequential relations between those repeating activities are obeyed, but cannot directly represent constraints for resource continuity [6], nor exploit the repetitive nature for the model itself. On the other hand, the Linear Scheduling Method (LSM) and closely related approaches with similar names, e.g. repetitive or location-based scheduling, which “analyze projects that are characterized as geometrically linear or repetitive in their operations” [14, p. 711], are able to take advantage of prevailing repetitiveness to derive a more intuitive model. As shown in Figure 1, such schedules can intuitively display how pairwise activities of unequal productivities are related in regards to their start and finish coordinates [15]. LSM was widely considered to be merely graphically-based scheduling that “is presented graphically as an \( X-Y \) plot where one axis represents [repeating] units, and the other time” [6, p. 270]. Such graphical, non-mathematical focus was a severe limitation of LSM and has hindered its computerization and broader application in the construction industry. However, that situation has changed through the introduction of the Productivity Scheduling Method (PSM), which employs singularity functions. They “provide a flexible and powerful mathematical model for construction activities and their buffers that are characterized by their linear or repetitive nature” [14, p. 711] to allow a comprehensive mathematically rigorous treatment. They support calendarized time axes [16] for more realistic models, which is desirable as it has been shown that “disregarding… calendars seems to be an oversimplification that can distort the results” [17, p. 404].

Lucko [18] described the steps of PSM as activity and buffer equations, initial stacking, minimum differences and differentiation for consolidation, which generates revised activity and buffer equations, and criticality analysis. Details of these steps are omitted in this paper for brevity. Stacking generates a feasible but extremely conservative and thus lengthy schedule, consolidation overlaps as many activities as possible within the sequence constraints. The approach of PSM is to convert all inputs about activity sequence, durations, leads or lags between activities, and
any time or work break into the mathematical model that is composed of singularity functions. The schedule (start and finish of each activity in the initial and final versions) and its criticality characteristics are outputs. Time \((y)\) is the independent variable in PSM per Equation 2, \(x\) is work, \(U_i\) is the number of repeating work units, \(D_i\) is the duration of activity \(i\), \(a_{pi}\) and \(a_{pi}^*\) are its start and finish, whose asterisk indicates that they may include a shifted start (prior delay \(d_{si}\) ) or extended duration (new delay \(d_{si}^*\)). Note that time is better measured on the \(y\)-axis, because it should be minimized by the algorithm. An analogous form with time as the dependent variable can be derived by switching the axes. An example with activities per Table 2 is shown in the \(x(y)\)-oriented linear schedule of Figure 1.

Figure 1 shows the schedule in a cumulative form, which means that the work quantity of each activity (vertical axis) increases with time that passes (horizontal axis). According to calculus, calculating the derivative of a function yields its rate of change [19]. Thus differentiating the cumulative Equation 2 returns the non-cumulative Equation 3. Figure 2 shows this non-cumulative form. Interestingly, the profile of non-cumulative activity equations is similar to a traditional bar chart. It differs from a bar chart only in that the vertical axis represents productivity and not just an activity name or label. It thus displays more information than a bar chart. The activity with the highest productivity is placed at the top, and the smaller the productivity the lower the height where activity bars are placed.

\[
x(y) = U_i \frac{d_{si}}{D_i + d_{si}} \left( y - a_{pi}^* \right)^v \left( y - a_{pi}^* \right) - \left( D_i + d_{si} \right) \left( y - a_{pi}^* \right)^v \left( y - a_{pi}^* \right)^0 \quad (2) \\
x'(y) = U_i \frac{d_{si}}{D_i + d_{si}} \left( y - a_{pi}^* \right)^v \left( y - a_{pi}^* \right) - \left( D_i + d_{si} \right) \left( y - a_{pi}^* \right)^v \left( y - a_{pi}^* \right)^0 \quad (3)
\]

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![Figure 1. Cumulative linear schedule for example.](image1)

![Figure 2. Non-cumulative linear schedule for example.](image2)

Overall, PSM with its systematic application of singularity functions provides a mathematical model that unifies compartmentalized concepts and project performance parameters, as used e.g. in minimizing the project duration and determining the criticality of activities under CPM, displaying graphically their start and finish dates as in bar charts, and representing the starts, finishes, sequence, and productivity of activities as under LSM. The cumulative and non-cumulative forms of the equations enhance scheduling research in that it becomes apparent that concepts whose relation was previously arduous to model can now be explicitly handled in an integrated quantitative manner.

3.2. Cash Flow Model (cumulative and non-cumulative forms)

The success or failure of construction projects strongly depends on cash flow management. Thus, modeling cash flows is a crucial problem in construction management. However, it is a thorny problem, because the interaction of cash outflows and inflows generates a zigzag-shaped balance that used to defy modeling attempts until recently [20].
Moreover, some phenomena related to cash flows exhibit a distinct periodicity [21], which should be modeled. Furthermore, Time Value of Money (TVM) can be considered explicitly [22]. Expanding the example from the previous section, assume that cost for each activity grows linearly [23]. Table 2 lists the cost \((C_i)\), markup \((M_i)\), and bill-to-pay-delay \((b_i)\) for each activity. Whereas the slope represented productivity in the linear schedule, the scale factor \((C_i / (D_i + d_i))\) in the singularity function per Equation 4 represents the rate of cost growth, which is the slope of the cost profile. Adding the markup to the cost yields a bill function per Equation 5. As bills are sent to the payer at the end of each period, an operator \(\lfloor \cdot \rfloor\) that rounds down the operand to its nearest integer is applied to the independent variable \(y\). Such rounding operator can easily express the “periodicity [that] occurs both in cash outflows and inflows that have a specific frequency and amplitude, e.g., overhead, billing, and payment functions” [20, p. 528]. The pay function per Equation 6 is derived from the bill function, but subtracts the bill-to-pay-delay \(b_i\) from each \(y\), which has the effect of moving the bill profile to the right to become the pay profile per Figure 3. Note that the cost, bill, pay, and balance profiles in Figure 3 model the entire project, adding the cash flows of the three activities accordingly. The balance is the difference between the sum of the cost functions (outflows) and the sum of all pay functions (inflows) per Equation 7. In a real project, \(b_i\) will be approximately 30 to 90 days; here it is assumed as one month. It is assumed that the balance function does not consider TVM. Its value depends on the period over which it is assessed, e.g. for financing interest or an unused credit fee, as analyzed in detail [20, 21, 22], but is excluded here for brevity.

Previous cash flow models are cumulative, yet the non-cumulative form of such cash flow models can also exist per Figure 4. Differentiating Equation 4 yields the non-cumulative cost function per Equation 8. A cumulative pay function has a stepped profile per Figure 3. Differentiating it would generate a bar chart-like profile, which would be incorrect, because the non-cumulative pay function should only be active at each pay time. Su and Lucko [21] solved this problem by introducing customized pay and signal functions for non-cumulative pay per Equations 9 and 10, which are spikes in Figure 4, while the difference between cost and pay is the dashed non-cumulative balance.

\[
\begin{align*}
\mathbf{z_{\text{cost}}}(y)_i &= \frac{C_i}{D_i + d_{2i}} \times \left[ (y - a_{\text{cost}}^i)^1 - (y - a_{\text{cost}}^i)^1 \right] \\
\mathbf{z_{\text{pay}}}(y)_i &= \frac{C_i \cdot (1 + M_i)}{D_i + d_{2i}} \times \left[ (y - b_i) - a_{\text{pay}}^i \right] - \left[(y - b_i) - a_{\text{pay}}^i \right] \\
\mathbf{z'_{\text{cost}}}(y)_i &= \frac{C_i}{D_i + d_{2i}} \times \left[ (y - a_{\text{cost}}^i)^0 - (y - a_{\text{cost}}^i)^0 \right] \\
\mathbf{z_{\text{balance}}}(y) &= \sum \mathbf{z_{\text{pay}}}(y)_i - \sum \mathbf{z_{\text{cost}}}(y)_i \\
\mathbf{z_{\text{each}_ \text{pay}}}(y) &= \frac{C_i \cdot (1 + M_i)}{D_i + d_{2i}} \times \mathbf{z_{\text{pay}_ \text{signal}}}(y) \\
\mathbf{z_{\text{pay}_ \text{signal}}}(y) &= \left[ (y - (a_{\text{pay}}^i + b_i)^i) - (y - (a_{\text{pay}}^i + b_i)^i) \right] - \left[ (y - (a_{\text{pay}}^i + b + 1)^i) - (y - (a_{\text{pay}}^i + b + 1)^i) \right] 
\end{align*}
\]
3.3. Resource Model (cumulative and non-cumulative forms)

The same approach as described for linear scheduling and cash flows can also be adopted for resource utilization. Its scale factor per Equation 11 represents the rate of resource consumption \( r \) [24], typically for general or specialized labor. Continuing the previous example, Table 2 lists the resource rate for each activity. Equations 11 and 12 express the cumulative and non-cumulative resource functions, respectively, which are shown in Figures 5 and 6.

Aligning these subsystems in their cumulative and non-cumulative forms fulfills Research Objective 2. Note that additional refinements can be added to reflect the aforementioned factors such as shifts \( d_1 \) (that move the start) and delays \( d_2 \) (that move the finish, i.e. expand duration) [14] to enhance its realism. This can be accomplished by adding them to the cutoff of the singularity function. Equation 13 links subsystems via the scaling factors of their pairwise variables for proportionality, notwithstanding that the cutoff in bill equations is further moved and rounded.

\[
\begin{align*}
\text{Figure 5. Cumulative resource profile for example.} \\
\text{Figure 6. Non-cumulative resource profile for example.}
\end{align*}
\]

\[
\begin{align*}
r(y) &= s_i \cdot \left( y - a_y \right)^4 - \left( y - a_y \right)^0 - (3 + d_2) \cdot \left( y - a_y \right)^0 \quad (11) \\
r'(y) &= s_i \cdot \left( y - a_y \right)^0 - \left( y - a_y \right)^0 \quad (12)
\end{align*}
\]

\[
\begin{align*}
f(x, y, z) &= \begin{cases} 
\frac{C_i}{U_i} x, & z = \frac{C_i}{D_i + d_{2i}} y, \quad x = \frac{U_i}{D_i + d_{2i}} y 
\end{cases} 
\end{align*}
\]

4. Interactions

The aforementioned models of schedule, cash flows, and resources exist as local subsystems of a construction project, where they focus on how a dimension of project performance develops in the short and long term. However, these subsystems suffer from the severe disadvantage that they become disconnected from the behaviour of the other dimensions, so that the informational power of experimenting with the impact of a change across these dimensions is lost. For example, a classic problem in scheduling is whether activities are treated as interruptible or continuous in their progress. Interruptability can reduce the total project duration in certain constellations by decreasing a lead / lag time between an activity and its predecessor and/or successor. However, the converse insight, that interruptability is not useful [25] but harmful, has been advocated as well, as an interruptible schedule jeopardizes resource continuity due to firing and hiring of labor with the associated problems of lacking motivation and low productivity. In experimenting with different strategies, one should always also monitor resource utilization in conjunction with the modifications to the schedule. Similar considerations apply to other interactions between multiple dimensions, each of which forms a sub-system. Holistically examining such global system therefore requires an integrated model.

4.1. Integrated Model for Multi-Objective Research

To integrate the aforementioned sub-systems in construction management, this research explores the dimensions of work (quantity), time, cost, and resource. It first explores three dimensions that progress concurrently and can still be plotted graphically. Selecting three of the possible combinations work-time-cost, work-time-resource,
time-cost-resource, and work-cost-resource, respectively. Table 3 lists the cumulative models for these four scenarios for individual activities per Table 2, which are then added to gain the model for the entire project. The trajectories of these models are plotted in the 3D coordinate systems of Figures 7 through 10. Subtracting the functions of payments and costs yields the balance \( f(x, y, z_{\text{cost}}) = f(x, y, z_{\text{pay}}) - f(x, y, z_{\text{cost}}) \) at any time. Note that in Figure 7 the curves \( f_{\text{cost}}, f_{\text{pay}}, \) and \( f_{\text{balance}} \) are the trajectories of individual activities in the 3D coordinate system, while projecting them e.g. onto the 2D time-cost plane yields the cost profile of each activity. Their sum is the trajectory of cost \( f_{\text{cost}} \) for the entire project. The pay and balance trajectories \( f_{\text{pay}} \) and \( f_{\text{balance}} \) in Figure 7 are shown for the entire project for clarity.

<table>
<thead>
<tr>
<th>Function</th>
<th>Projection onto x-z-Plane</th>
<th>Projection onto y-z-Plane</th>
<th>Projection onto x-y-Plane</th>
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<tbody>
<tr>
<td>( 7, (11) ) [ f(x, y, z_{\text{cost}}) ] [ \frac{C_i}{U_i} \cdot x ] [ \frac{C_i}{D_i + d_{2i}} \cdot y ] [ x = \frac{U_i}{D_i + d_{2i}} \cdot y ]</td>
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<td>( 7, (11) ) [ f(x, y, z_{\text{pay}}) ] [ \frac{C_i}{U_i} \cdot x ] [ \frac{C_i}{D_i + d_{2i}} \cdot y ] [ x = \frac{U_i}{D_i + d_{2i}} \cdot y ]</td>
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<td>( 8, (12) ) [ f(x, y, r) ] [ r = \frac{s}{D_i + d_{2i}} \cdot \frac{U_i}{y} ] [ r = \frac{s}{D_i + d_{2i}} \cdot y ] [ x = \frac{U_i}{D_i + d_{2i}} \cdot y ]</td>
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<td>( 9, (13) ) [ f(z_{\text{cost}}, y, r) ] [ r = \frac{s}{C_i} \cdot \frac{D_i + d_{2i}}{z_{\text{cost}}} ] [ r = \frac{s}{D_i + d_{2i}} \cdot y ] [ y = \frac{C_i}{D_i + d_{2i}} \cdot y ]</td>
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<td>( 10, (14) ) [ f(x, z_{\text{cost}}, r) ] [ r = \frac{s}{C_i} \cdot \frac{D_i + d_{2i}}{U_i} \cdot x ] [ r = \frac{s}{C_i} \cdot \frac{D_i + d_{2i}}{U_i} \cdot y ] [ y = \frac{C_i}{U_i} \cdot x ]</td>
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An important feature of this 3D trajectory is that projecting it onto any 2D plane yields the desired plot of the sub-system without decreasing accuracy. For example, the projection onto the time-cost plane of Figure 7 is identical to the cash flow profile per Figure 3, while the projection onto the work-time plane is the linear schedule per Figure 1, and the third possible projection, which appears not to have been explored individually, is the profile on the work-cost plane. Since work typically must be finished sequentially, it grows chronologically, which implicitly exhibits the relation of time and cost. This projection onto a work-cost plane tracks how much cost is incurred by finishing work. It has potential use in earned value analysis [26], whose indices of budgeted cost of work performed (BCWP), budgeted cost of work scheduled (BCWS), and actual cost of work performed (ACWP) track the progress.

A virtue of 3D trajectories is that they can reveal the otherwise hidden impacts of changes within a traditional 2D relationship, i.e. a ‘dragging effect’ that pulls the profile of work and time into the third dimension by the respective cost for each point. Newly added depth that a formerly 2D profile gains, which is visible throughout the 3D figures, thus can be a valuable tool to illustrate the degree of interactions among multiple dimensions within the same project.

Figures 8, 9, and 10 are generally similar to Figure 7, but plotted in different three dimensional systems. Figure 8 exhibits the trajectories of each activity and their combined effect in a work-time-resource system. Projections onto a work-time or time-resource 2D plane will return the linear schedule per Figure 1 or resource profile per Figure 5. The combined curve displays the growth path of the project, on which each point informs how much work quantity has been finished at what time by using how many worker-hours. Replacing work by cost dimension yields Figure 9, in which the combined curve informs how much cost was incurred at what time by using how many worker-hours. The final possible combination (work-cost-resource) per Figure 10 differs from the other three figures. Time is merely expressed implicitly, because work, cost, and resource values progress from zero to their maximum amounts. Whichever the chosen scenario, the integrated model can be mathematically expressed as singularity functions and plotted in a 3D coordinate system. Since the integrated model is ‘grown’ – or rather assembled – from sub-system models, all details from these sub-systems can be preserved. By modifying parameters in any sub-system, it allows to monitor the global effects directly. Sensitivity analyses and optimization algorithms would generate or handle
growth paths in form of a trajectory bundle through the project space in the 3D coordinate system. This approach
fills the modeling gap between the two aforementioned steps, which may improve the efficiency of such algorithms.

Figure 7. Cumulative work-time-cost trajectory.  
Figure 8. Cumulative work-time-resource trajectory.

Figure 9. Cumulative time-cost-resource trajectory.  
Figure 10. Cumulative work-cost-resource trajectory.

Differentiating the cumulative equations of Table 3 yields non-cumulative ones, which are plotted in Figures 11
through 14 as individual and total curves. Again, they can be projected onto 2D planes, which yields Figures 2 and 4.
Note that spikes within Figure 11 are separate payments, which reach above the plane that is defined by \( cost = 0 \),
and that non-cumulative balances can be negative or positive. Again, dragging a 2D profile into the third dimension
reveals the relative influence of that new performance parameter. Figure 12 displays three activities bars \( f_A, f_B, \) and \( f_C \)
by using the differentiated form of the respective function per Table 3. Adding them yields the total trajectory \( f_{total} \).
which reveals the absolute productivity and resource rate at any time. Figure 13 resembles Figure 12; it shows three activity bars and charts the absolute productivity and cost rate at any time. Interesting are the projections onto the resource rate-productivity plane in Figure 12 and onto the resource rate-cost rate plane in Figure 13. The former informs the project manager how productive the workers are in absolute terms, while the latter informs how cost-intensive workers are (assuming that only direct costs, e.g. wages, are tracked for simplicity of this example).

All of the growth paths in Figures 11 through 13 start from the plane that is defined by \( t = 0 \). An exception is Figure 14, which omits the time dimension to plot a productivity-cost rate-resource rate coordinate system. Its three trajectories display the differentiated form of the respective function per Table 3. They track how much productivity is achieved for specific resource rates and cost rates. Similar to Figure 10, which implied the time dimension, the growth path in Figure 14 can be determined by the sequence of activities. It starts from the end of \( f_A \) (resource rate = 10), climbs to resource rate = 30, drops to the end of \( f_B \) (resource rate = 20), then climbs again to resource rate = 30, drops to the end of \( f_C \) (resource rate = 10), and finally returns to the origin. This undulation of the growth path can be verified via Figures 12 and 13. Non-cumulative trajectories can thus provide valuable information about a project.

4.2. Singularity Functions Enabling nD Models

An intriguing theoretical possibility exists that adding further dimensions to the aforementioned integrated model will generate 4D and eventually \( n \)-D models. This will create additional interfaces to reflect the interactions between project performance parameters. While such models are impossible to display graphically, they are easily expressed mathematically. Singularity functions can be used to model and explore complex behaviors in multiple dimensions, which fulfills Research Objective 3 and offers a novel avenue toward holistic quantitative project management.

5. Contributions and Recommendations

Traditional construction management has compartmentalized the treatment of projects into sub-systems, including time, cost, and resources. However, such approaches have strongly simplified or even omitted other dimensions in their models, and has created the conundrum that optimization approaches even for multi-objective problems may tend toward identifying any local optimum instead of the global optimum. Numerous algorithms are described in the literature that have investigated the performance of such algorithms. Most simplified sub-systems models, perhaps misunderstanding of Ockham’s razor, who can be paraphrased as having advocated including ‘as much as necessary, as little as possible’ of the complexity of the real world in a model. Thereby accuracy and realism may be sacrificed for the sake of a supposedly efficient model, albeit one that may produce somewhat misleading results. However, accuracy and realism need not be contradictory goals to the efficiency of models. This study addresses this challenge by employing singularity functions in a newly integrated manner, which can be customized as needed by the user.

Contributions to the body of knowledge are as follows: First, having aligned 2D models of project performance parameters, including linear schedules, cash flows, and resource histograms, which share a similar mathematical structure when expressed as singularity functions. Second, having explored the interactions in the cumulative and non-cumulative forms of the sub-system models, which is enabled by sharing a time axis, either explicitly or at least implicitly. Non-cumulative forms are derived from cumulative ones to measure the rate of change of such project performance parameter. Third, integrating them into 3D models, which provide an information-rich representation of a project, and can theoretically be extended to higher orders by incorporating additional quantitative dimensions.

Integrating familiar graphical representations that are used in construction management like bar charts, linear schedules, cash flow profiles, and resource histograms is likely to facilitate its adoption into professional practice. The versatility of the new approach is evident in that combinations of three out of four dimensions (work, time, cost, and resources) in the cumulative or non-cumulative forms all communicate managerially relevant information and allow a ‘depth’ of exploration that has been lacking to date. This approach fosters perceiving a project as a multi-dimensional growing endeavor, whose trajectory moves from the origin through the project space in the coordinate system. At any point, the behavior in each of its various dimensions is preserved in detail and their interactions can be explored as needed. Further research is recommended to explore several aspects that will enhance the new model:
• Deriving an algorithm to transform the ubiquitous network-based CPM schedules, which are widely used in the construction industry, into linear schedules with singularity functions for direct integration into the new model;
• Incorporating additional features and constraints, e.g. interruptible activities, to extend its analytical capabilities;
• Quantifying the benefit of not diluting its level of detail when adding dimensions to the model and measuring the computational efficiency of multi-objective optimization algorithms that are applied to singularity functions.

Figure 11. Non-cumulative work-time-cost trajectory.
Figure 12. Non-cumulative work-time-resource trajectory.
Figure 13. Non-cumulative time-cost-resource trajectory.
Figure 14. Non-cumulative work-cost-resource trajectory.
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