Supporting Financial Decision-Making Based on Time Value of Money with Singularity Functions in Cash Flow Models

Abstract

Existing budgeting approaches differ in whether or not they consider the Time Value of Money. A novel use of singularity functions in construction management has the potential to enhance cash flows models toward maximizing their net present value. This type of function can model a complete schedule, which serves as the underlying timeline for all financial transactions. Their variable amounts and constraints are expressed by singularity functions, converted from costs via bills into payments, and compounded toward the overall net present value for financial decision-making. Contributions to the body of knowledge include deriving exact interest on variable balances for any duration, creating a valuation algorithm, and exploring how the uneven solution space that cash flow profiles create can be searched successfully with a genetic algorithm.

Keywords: Financial management, cash flows, singularity functions, optimization
Introduction

Construction projects should be carefully planned and managed in their financial aspects and ultimately must create value to support the survival of the company and support its long-term growth. Any company is vulnerable to the threat of bankruptcy – insufficient cash flows – that could halt its otherwise timely and safe operations. Cash flows of its construction projects should therefore receive full attention. They must be optimized within their constraint framework that is set by a schedule that comprises many cost-inducing activities of productive or administrative nature, milestones, and deadlines; and a budget with its financial terms and conditions that are specific to the market, the company, or its planned project, e.g. available credit and interest rates.

However, cash flows themselves have proven to be notoriously difficult to be modeled in an integrated manner, as their individual elements are of strongly different nature; some of them are fixed while others are variable, conditional, or even cursive. Moreover, while profit is easily understood and commonly used by contractors as a financial measure of project performance, it neglects the impact of the Time Value of Money – the phenomenon that funds are worth unequal amounts, depending on when they are assessed. Previous studies that examined cash flows for construction management applications focused too narrowly on maximizing profit, instead of considering Time Value of Money. Furthermore, such models were typically based on network schedules that model only time, rather than linear schedules that also express planned and actual work. Those models generally exhibited an inability to accurately model the varying balances of cash flows in a construction project with an integrated expression. This challenge is attenuated by the fact that certain cost elements are conditional. Financial decision-making must therefore be equipped with an improved tool to provide such needed modeling and analytical capabilities.
Singularity functions, defined via a case distinction of ranges where an expression is non-zero, show significant potential to model the quantitative aspects of construction projects. Recent research has focused on developing algorithms to calculate lesser-known linear schedules, which display work over time. The logical next step is to investigate how singularity functions can also capture the intricacies of financial phenomena to benefit cash flow management of construction projects to maximize their value. The following sections review existing approaches to establish the need and objectives in detail, define singularity functions, explain the important elements of cash flows and anchor them in a linear schedule, derive their variable balances over time, express dependencies between activities, and assemble the overall model. A genetic algorithm is selected to optimize its overall net present value and the new approach is validated in several calculations.

Importance of Time Value of Money for Decision-Making

Most construction projects are performed over a course of several months to several years. Their owner, general contractor, and subcontractors all commit to a long-term endeavor that becomes a significant part of the financial side of their business enterprise. While the focus of this paper lies on cash flows from the contractor’s perspective, its approach can also be of interest to owners, who must decide if an investment to generate later possible revenues should be undertaken. Such long-term projects incur various time-dependent financial phenomena: Contractors must borrow money to finance projects until progress payments bring them into the profit zone (Warszawski 2003), as advance payments from the owner that offset early expenses like mobilization are uncommon in commercial construction (unlike small maintenance and repair jobs that require a deposit). The duration of financing depends on the cost distribution within a project; it may range from only the first months to having to support most operations. Contractors will always incur a
financing cost, here represented by a generic interest rate \( i \), whose exact nature, composition, and value depends on various factors. They include its corporate structure, financial situation, and relations with external funders, and the mechanisms that are chosen for financing. The latter is a mixture from among liquid capital from retained earnings or selling assets, debt obligations of bank loans or issuing bonds, or possibly selling ownership shares. For publicly traded companies its burden may be captured by the weighted average cost of capital (Lazonick and O’Brien). It would also be compared with foregoing alternative revenue-generating investments that could yield a return rate to consider their marginal cost of capital. In any case, costs are compounded over multiple time periods, typically months, per the \textit{Time Value of Money} concept until the balance returns to zero. Interactions within a project form a complex system (Cui \textit{et al.} 2010) so that any timing of schedule activities and events impacts its value. In turn, this impacts \textit{profit}, which various studies used as the objective function for optimization, surprisingly ignoring the Time Value of Money. An additional time-dependent phenomenon, inflation, causes future costs to rise and payments to lose value and acts similarly as interest upon cash flows of contractors (Ranasinghe 1996), so that it could be modeled in analogy to the generic interest component.

\textbf{Shortcomings of Existing Models}

A diverse set of measures exist in capital budgeting, the area of finance and business economics that assesses investment opportunities of required single or multiple expenses (cash outflows) to create future revenues (cash inflows) that are anticipated to be more valuable. Among the well-known measures are the profit, profitability index, normal and discounted payback periods, net present value (NPV), net uniform series (equivalent annual annuity), internal (discounted) rate of return (IRR), modified internal rate of return (MIRR, marginal return on invested capital), and
(average) accounting rate of return (ARR) (Ross et al. 2000, Kennedy and Plath 1994, Au 1988), whose commonly accepted definitions and fundamental assumptions are presented in Table 1.

<Insert Table 1 here>

Measures can be categorized by whether or not they consider the Time Value of Money. Much literature cautions against using those that ignore it because of their vast simplifications (Park et al. 2009), but they remain popular in business practice due to their simple calculation (Graham and Harvey 2001). Commonly noted challenges are that (a) all Time Value of Money measures require forecasting a discounting rate (for the present value) that strongly influences their values (McDaniel et al. 1988), the MIRR additionally requires a compounding rate for future values of reinvested inflows over n periods (Shull 1992); (b) both payback period methods approximate a ‘break-even’ time by interpolation, but omit any potential cash flows beyond it (Ross et al. 2000, Longmore 1989); (c) the profitability index, payback periods, and IRR implicitly assume only a conventional cash flow pattern (Zhang 2005), i.e. one outflow followed by many small inflows, as multiple sign changes in the balance can create multiple (invalid) solution values; (d) the IRR assumes that all inflows can be reinvested at the same rate, requires users to select a ‘hurdle rate’ of acceptance, and cannot be solved directly, only iteratively (Hajdasinski 2004); and (e) the ARR must assume a specific depreciation type and duration for its average book value and was criticized for considering accounting profit rather than actual cash flow values (Ross et al. 2000).

The focus therefore is placed on the NPV as the primary theoretically valid way to value the cash flows per their Time Value of Money. Its importance is underlined by the fact that many companies reluctantly treat financing and its impact on profit as a cost of doing business (Setzer
2009) and thus may not optimize it explicitly. Moreover, while billing cycles may be perceived as too short to consider Time Value of Money, the tendency in the industry is that inflows incur significant delays, as “the modern businessman ... is quick to bill but slow to pay” (Berlekamp et al. 2001, p. 126), impacting especially subcontractors. Furthermore, the issue is attenuated by the uncertainty of future events and the often-cited tight profit margins that may exist in construction projects, creating an urgent need for optimizing the financial performance of projects in detail, e.g. by front-loading expenses and other strategies if possible and permissible (Cui et al. 2010).

**Need for New Cash Flow Model**

Research on financial planning created various cash flow models, including spreadsheets (Cui et al. 2010, Hegazy and Ersahin 2001). Cash flows were acknowledged to be particularly difficult to model (Chen et al. 2005, Kenley 2003) due to the complexity that arises from incorporating very disparate elements: They “differ in source and recipient; constraint conditions; whether they occur uniquely, regularly, or even repetitively; whether they accrue in bulk or incrementally; whether their timing determines their amount; and whether they depend on previous payments” (Lucko 2011b, p. 523). Only few studies discounted in their approach, e.g. Ammar (2011, p. 66) explored time-cost tradeoffs and noted that “most of existing techniques... do not consider time value of money.” Chen et al. (2010) used ant-colony optimization for a resource-constrained schedule; Pinder and Maruchek (1996) evaluated heuristic rules. Such studies employed network schedules, but those suffer from the conceptual weakness of being “effectively one-dimensional” (Lucko 2007, p. 2159), displaying only activity starts, durations, and finishes. This conceals their actual inputs of work amount and production rate. Moreover, they cannot easily prevent conflicts
between activities (Lucko 2008). Linear scheduling is therefore preferred in this research as is described later. Numerous other recent studies completely ignored the Time Value of Money:


Statements on the importance of optimizing cash flows rather than mere profits permeate the financial literature (Reider and Heyler 2003). Several well-known business cases illustrated that companies might post accounting profits while poor cash flows in fact brought them close to bankruptcy (Howell 2002, Fisher and McGowan 1983). Financial models and budgeting must therefore urgently be reformed (Bourne et al. 2002). The aforementioned limitations of existing models create a need for a new cash flow modeling approach to support the financial decision-making by planners of construction projects. It will be addressed in four Research Objectives:

- Establish a direct conversion between elements of network schedules and linear schedules;
- Examine balances that occur over partial periods and derive new equation for financing fees;
- Aggregate variable cash flows while considering dependencies and Time Value of Money;
- Integrate singularity function model and perform its validation with a complete optimization.
Singularity Functions Definition

Singularity functions generalize traditional mathematical functions because their operators act like switches to activate their components. Their unique ability to model complex behaviors with singularities, as the name implies, i.e. discontinuities in value, slope, or curvature, means that they can be easily customized to any user’s needs. Originally used by structural engineers to calculate shear and moment distributions along beams under complex loads (Macaulay 1919), this versatile family of functions provides the ideal tool to accurately model cash flows that consider the Time Value of Money. They are composed by adding basic terms of Equation 1:

\[ z(y) = s \cdot (y - a)^n \]  

(1)

where \( z \) and \( y \) are the dependent and independent variables, here cost and time respectively, \( s \) is the strength factor for \( z(y) \) that may contain additional operators if desired, \( a \) is the cutoff on the \( y \)-axis when the term becomes active and gives non-zero values, and the exponent \( n \) determines the behavior. This basic term can be integrated and differentiated into \( 1/(n+1) \cdot s \cdot (y - a)^{n+1} \) and \( n \cdot s \cdot (y - a)^{n-1} \) akin to traditional functions. Composing a singularity function from multiple basic terms as shown in Figure 1, which contains step and shift singularities, should apply three important rules that together ensure a consistent structure for clarity and brevity (Lucko 2011a):

- **Superposition**: Basic terms remain active until positive infinity on the \( y \)-axis. Each expresses exactly one change in its exponential order. Complex behavior is modeled cumulatively by overlaying as many basic terms of different \( s, n, \) and \( a \) as needed. After becoming active at their respective \( a \), their sum gives the desired customized total \( z(y) \). Deactivating a basic term after a range on the \( y \)-axis is done by adding a negative term of the same \( s \) and \( n \), but later \( a \).
• **Sorting:** It is recommended to emulate the standard format of polynomial functions, i.e. sort all basic terms from left to right in a singularity function by their cutoff $a$, then by decreasing exponent $n$, and then by decreasing $s$, if applicable. Such consistency improves readability.

• **Simplification:** After sorting has been performed, multiple basic terms with the same $a$ and $n$ should be simplified by adding their values of $s$, which shortens long singularity functions.

The following sections explain how these functions model linear schedules and their cash flows.

<Insert Figure 1 here>

**Linear Schedule Model**

A schedule is necessary to provide a common timeframe to which all time-related elements that are used in the subsequent analysis refer. In the following, the term ‘activity’ can therefore mean any productive or administrative process with a non-zero duration or instantaneous event. For simplification, all costs are tied to such activities; it is acknowledged that transactions on the cost side, e.g. from materials purchases, may precede or succeed a construction operation in reality.

Considering the dominance of network schedules, linear scheduling is a very versatile yet underused construction management technique (Lucko 2009). Each activity in linear schedules is diagrammed as a piecewise segmental curve whose slope is proportional to its productivity. In other words, it explicitly expressed the work amount that is produced over its duration. In such schedule, possible conflicts between adjacent activities are avoided by enforcing their minimum proximity with buffers, underperforming activities are easily identified, and overall performance can be improved by aligning the individual productivities of all of its activities as far as possible.
Singularity functions have been successfully applied to analyzing linear schedules (Lucko 2009). While various early studies had assumed a constant productivity for each activity (as do network schedules by default), newer approaches indeed consider their variable nature (Duffy et al. 2011). Such progress plans or actual measurement can be modeled with singularity functions at any desired resolution for both time and work units. In effect, activities will thus consist of any number of segments as is appropriate for the desired level of detail. One singularity function is written per activity and sequentially added to the linear schedule, whose total duration will then be minimized by reducing interstitial gaps between adjacent activities. Details on that scheduling algorithm can be found in a previous publication (Lucko 2009) and are omitted here for brevity.

**Converting Network Schedules into Linear Schedules**

Singularity functions can express linear schedules as well as their cash flows (Lucko 2011b), which simplifies their joint analysis and optimization. However, users may wish to alternatively use network schedules due to familiarity, which were used in the subsequent cash flow example as originally published (Elazouni and Metwally 2005). **Research Objective 1** is therefore to derive a conversion of networks and linear schedules. It assumes that network activities complete a generic work amount $W = 1.0$ units $[u]$ (100%) over their duration $D$ from start $a_S$ to finish $a_F = a_S + D$. Start dates from a network schedule simply become the intercepts in the linear schedule when their activity equations cross the time axis in Equation 2. It demonstrates the equivalence of expressing work $x$ as a function of time $y$ or vice versa. Possible changes within activities, e.g. from productivity changes or interruptions, can only be visualized by a linear schedule, which is the reason for employing it in the subsequent modeling. This would mean analogously applying Equation 2 to individual activity segments that produce the respective range of work $u_S$ to $u_F$. 

URL: http://mc.manuscriptcentral.com/rcme
Cash Outflows

Each construction project experiences numerous cash outflows from its ongoing costs to pay for workers’ salaries and fringe benefits, the purchase, delivery, storage, and installation of materials and assemblies, consumables and tools, and owning or renting and operating heavy equipment. Other costs are mobilizing and demobilizing site installations and paying overhead for the home office. Administrative costs include tax, bond, insurance, and permits. Financing costs accrue in a unique manner because they are commonly (a) based on a variable balance, (b) assessed only at specific periods, (c) compounded over time, and (d) subject to a credit limit. Liquidated damages or any penalties are subject to the condition of late completion. Conversely, bonuses may reward early completion. Even other types of conditional cost elements can be expressed by using binary decision variables. Together, such transactions create a complex behavior (Cui et al. 2010) that requires an integrated model of the various cash flow elements. Most researchers assumed costs as linearly distributed over activities and a simple interest rate in compounding or discounting (Abido and Elazouni 2011), which is retained in the current study. This is primarily true for budget planning and optimization purposes when the exact timings of the many incremental transactions, such as purchase orders for materials, are still open. Yet singularity function could also model more detailed or nonlinear behaviors, which is excluded from this scope for brevity.

Modeling Variable Balances

Published models in the literature always assume changes only at periods, not within them (e.g. Newnan et al. 2004). In other words, such calculations allow only integer values of $n$ as points in

$$x(y)_{work} = W/D \cdot \left( y - a_s \right)^i - \left( y - a_f \right)^i \quad \Leftrightarrow \quad y(x)_{time} = D/W \cdot \left( x - 0.0u \right)^i - \left( x - 1.0u \right)^i$$

(2)
time when activities start and finish, interest is assessed, and payments occur, not fractional ones. However, realistic construction activities need not only start or finish exactly at period cutoffs – typically monthly – but may do so on any workday. Moreover, overlapping the ongoing costs of multiple concurrent activities and monthly progress payments can create fractional points in time when the overall balance changes from a positive to a negative value or vice versa. Furthermore, previous studies used variable balances, but simplified the interest (Elazouni and Metwally 2005, Tanchoco et al. 1981) to the disadvantage of contractors (Lucko and Thompson 2010). Research Objective 2 thus is to derive a formula to accurately calculate such financing. This new equation is inspired by an annuity, i.e. a regular series of payments of equal amounts over a limited range of time. Each payment \( pmt \) creates its own interest stream that is compounded by \( FV_{pmt} = PV \cdot (1 + i)^p \), where \( FV \) and \( PV \) are the future and present values, \( i \) is the interest rate per period, and \( p \) is the number of periods. The annuity formula takes the difference of two versions of the interest streams, one of which is shifted by one time unit. Thus all terms cancel each other out, except for the very first and last (Newnan et al. 2004). This collapsing effect resembles a telescoping series (Lucko and Thompson 2010) with pairwise positive and negative operands. It yields \( FV_{ann} = A \cdot \frac{[(1 + i)^p - 1]}{i} \), where \( A \) is the payment \( (ann) \) that is repeated at the ends of integer time periods.

Deriving the new equation for the future value of a variable balance \( (bal) \) analogously employs incremental payments, but shortens each partial period \( 1/k \) infinitesimally toward zero. The sum of all terms asymptotically approaches a limit that is the desired result. It is assumed that the start \( a_S \) and finish \( a_F \) of the activity that requires financing can occur at any point in time between integer periods as shown in Figure 2. Each incremental payment accrues exponential interest over a partial duration that extends to the integer point in time at or directly following \( a_F \).
The following series are sorted by decreasing partial duration. In Equation 3 said durations range from $D + \delta$ to only $\delta$, where $D$ abbreviates $a_F - a_S$ and $\delta$ is the partial period from $a_F$ to the next integer time, $\lceil a_F \rceil - a_F$. Its roundup operator $\lceil \, \rceil$ was introduced with its counterpart $\lfloor \, \rceil$ by Iverson (1962). Together these many interest streams compose a finite geometric series, as “each term is the $(1 + i)^{1/k}$–fold of the previous one” (Lucko and Thompson 2010, p. 3043). Equation 4 applies that factor here to shift all terms of the series by $1/k$, from which the original version is then subtracted in Equation 5. This leaves only the first and last terms. Equation 6 divides to isolate the common factor $FV_{bal}$ and applies limit operators to both the dividend and divisor, as the limit of such ratio is the ratio of their limits. As the number of infinitesimally short periods $k$ approaches infinity, $1/k$ approaches zero and will vanish from the dividend. Per Equation 7 the divisor series receives the temporary substitutions $h = 1/k$ and $\nu = 1 + i$, is extended by $\nu^x$, and re-substituted after simplification. The mathematical proof converges to the natural logarithm $\ln(1 + i)$. Its definition as the limit of infinitesimal intervals $h$ (Jacobson and Chinn 1968) is shorter than the well-known approximation for it by an infinite polynomial Taylor series (Weisstein 2012).

\[
FV_{bal} = A/k \cdot \left( (1+i)^{D+\delta} + (1+i)^{D+\delta} \cdot \frac{1}{k} + \ldots + (1+i)^{D+\delta} \cdot \frac{(D-1)}{k} + (1+i)^{\delta} \right) \tag{3}
\]

\[
FV_{bal} \cdot (1+i)^{-1} = A/k \cdot \left( (1+i)^{D+\delta} \cdot \frac{1}{k} + (1+i)^{D+\delta} \cdot \frac{1}{k} + \ldots + (1+i)^{D+\delta} \cdot \frac{(D-1)}{k} \right) \tag{4}
\]

\[
FV_{bal} \cdot \left( (1+i)^{-1} - 1 \right) = A/k \cdot \left( (1+i)^{D+\delta} \cdot \frac{1}{k} - (1+i)^{\nu} \right) \tag{5}
\]

\[
FV_{bal} = A \cdot \lim_{k \to \infty} \left[ (1+i)^{D+\delta} \cdot \frac{1}{k} - (1+i)^{\nu} \right] \bigg/ \lim_{k \to \infty} \left[ (1+i)^{-1} - 1 \right] \tag{6}
\]
\[
\lim_{k \to \infty} k \cdot \left[ \left(1 + \frac{1}{i} \right)^{\frac{k}{h}} - 1 \right] = \lim_{k \to \infty} \frac{\nu^k - 1}{h} = \frac{1}{\nu^x} \cdot \lim_{k \to \infty} \frac{\nu^{x+k} - \nu^x}{h} = \frac{1}{\nu^x} \frac{d}{dx} \nu^x = \frac{1}{\nu^x} \left[ \nu^x \cdot \ln(\nu) \frac{dx}{dx} \right] = \ln(\nu) = \ln(1+i)
\]

(7)

\[FV_{\text{bal}} = A \cdot \left[ (1+i)^{D+\delta} - (1+i)^{\delta} \right] / \ln(1+i) \]

(8)

The final new Equation 8 includes interest streams that are assessed at each integer point in time, e.g. at \( t = 2, 3, \) and 4 and all further times until the remaining balance is fully paid off. In special cases it simplifies further: Its dividend becomes \((1+i)^D - 1\) if a balance grows only in the second half of one period, versus \(1 + i - (1+i)^\delta\) if it grows in the first half and then remains constant. A linearly growing balance across one period gives \(FV = A \cdot i / \ln(1+i)\) (Lucko and Thompson 2010). Further extensions can be derived if \(A\) undergoes growth, but are omitted here for brevity. Its net present value is then obtained by discounting it by the factor \((1+i)^{a_F-a_S}\) to time zero.

**Modeling Variable Execution**

Equation 9 transforms Equation 2, work over time, into its new form, cost over time (cost slope), by introducing the total cost \(C\) and the tax rate \(T\). To rigorously test the capabilities of this model in the subsequent optimization both shifts and delays may occur, the latter of which had not been modeled in prior studies (Lucko 2011b, Elazouni and Metwally 2005). Thus two new terms model that an activity may vary in its start and finish dates, i.e. a shift \(d_1\) in both \(a_S\) and \(a_F\), or in its duration, i.e. a delay \(d_2\) only in \(a_F\). They can have internal or external causes, e.g. a ripple effect of earlier delays, a local crew unavailability or equipment breakdown, or sudden adverse weather. In other words, \(d_1\) shifts the execution and \(d_2\) reduces productivity and stretches costs over the longer duration. If costs change independently of duration, \(C\) could be modified as well.
In the remainder of the paper an asterisk will indicate that a variable includes shifts and/or delays for flexibility, with \[ \delta^* = [a_F + d_1 + d_2] - (a_F + d_1 + d_2), \ a_s^* = a_s + d_1, \text{ and } a_F^* = a_F + d_1 + d_2. \]

\[ z(y)_{\text{const}} = (1 + T) \cdot C/(D + d_2) \cdot \left[ (y - a_s - d_1)' - (y - a_F - d_1 - d_2)' \right] \quad (9) \]

**Modeling Dependencies**

To generalize fixed links between starts and finishes in network schedules, dependencies within linear schedules are modeled per **Research Objective 3** by carefully evaluating the sequence of activities, buffers, and other constraints. Attempts to capture such constraints with ‘soft logic’ suffered from an overly narrow distinction into sequential versus parallel execution (Tamimi and Diekmann 1988); in fact, both can occur for two overlapping activities as shown on the right side of Figure 3. Moreover, even applying soft logic to multiple repetitive subtasks (Fan and Tserng 2006) or extending it akin to applying Boolean logic (Wang 2005) ignored that relationships can exist continuously, not just between starts and finishes. Furthermore, the duration between the minimum calculated project finish (but before optimizing its cash flows considering the Time Value of Money) and the contractually specified deadline provides a ‘project float’ (O’Brien and Plotnick 2006) that in some cases may allow even critical activities, e.g. \( B \) and \( E \) on the left side of Figure 3, to be shifted during an optimization. It is here abbreviated \( e \) for extension. The five relationships \( A-D, A-F, A-C, B-C, \) and \( B-E \) apply to any points of equal work amounts within these activity pairs to ensure that a successor can never overtake its predecessors, even if it has a higher productivity. Implementing such dependencies for all linear schedules requires a modified approach. In it, shift \( d_1 \) and delay \( d_2 \) values are randomized for all activities by sequence step and within it for \( d_1 \) before \( d_2 \). Here one could even incorporate unique probabilities for each activity of incurring \( d_1 \) and/or \( d_2 \). In each step, the linear schedule is updated and only possible values for
yet-to-be-randomized $d_1$ and $d_2$ are active. This avoids the problem of infeasible options arising in the optimization; which had previously required a ‘repair’ (Elazouni and Metwally 2005).

<Insert Figure 3 here>

Cash Inflows

Discounting with Singularity Functions

Existing cash flow models have omitted discounting, which assesses the value of payments of different amounts and times (and even different growing behaviors) at any common point in time for decision-making. Single payments $z(y)_{pmu} = c \cdot (y-a)^0$, where $c$ is any increment of money, are discounted by Equation 10 over their time $a$ to equivalent net present values at time $y = 0$ and then added to give the total net present value of all cash flows. Depending on the financing terms for a contractor, the model could distinguish different interest rates for financing the outflows or investing the inflows, for example depositing them into interest-bearing business bank accounts.

$$z(y)_{npv} = (1+i)^{-a} \cdot c \cdot (y-0)^0$$ (10)

Converting Growth into Equivalent Payments

Any growing partial debt as shown in Figure 2 represents distributed cost over time. It must be converted into an equivalent single payment to be processed correctly in the subsequent analysis while considering the Time Value of Money. The question of equivalency raises two questions, its timing and amount. Assuming an example of debt growing from $0$ to $100$ within one period at $r = 5\%$ interest per period, the interest alone per Equation 8 is $100 \cdot [(0.05 / ln(1 + 0.05)) - 1]$ = $2.47967157$, which correctly is slightly less than interest on an average constant balance of
$50 \cdot 0.05 = $2.50. Equivalency of a variable balance and a single payment could be interpreted to mean that the same interest would accrue. However, this interpretation is incorrect: An interest of $2.47967157 accrues on a constant initial balance of $2.47967157 / 0.05 = $49.593431. But its balance including interest of $52.073103 at \( y = 1 \) yields incorrect interest in all future periods. Therefore, equivalency correctly means that the same balance is reached at \( y = 1 \), which is solved as $102.47967157 / 1.05 = $97.599687 as a constant balance at \( y = 0 \). This interpretation is used in this paper because it preserves the correctness of financing costs and underlines the significant impact of the Time Value of Money. The approach converts any growing balance into a single equivalent lump sum payment at time \( \lceil a_F^* \rceil + b \) that is then discounted to \( y = 0 \) per Equation 10.

This approach of converting variable balances into single payments that are discounted to their NPV could naturally also be applied to any positive balances that could earn interest at the rate \( i_2 \) over any partial periods. For brevity, the option is not included in the subsequent example.

Whereas cash outflows are assumed linearly growing, the related case of stepwise growth occurs for cash inflows and is modeled by Equation 11, which uses the rounddown operator \( \lfloor \rfloor \). For activities that extend over several periods their individual progress payments have largely constant values, i.e. take the form of annuities. While financial literature provides a formula for the future value of an annuity as explained previously, that formula only works at integer periods \( p \) themselves, but fails to give correct values at any partial periods, e.g. for \( p = 0.5 \) and an assumed \( i = 5\% \) it would give the value \( A \cdot 0.49390153 \), even though no annuity has been paid yet; they only start at the end of the first period! An improved stepwise equation must therefore be derived, which again uses the ‘telescoping’ approach, where \( b \) is the billing-to-payment delay. Equation 12 for the FV comprises a regular stepped part plus partial first and last payments if the activity starts and/or finishes at non-integer times, which Equation 13 discounts to the origin.
\begin{align}
  z(y)_{\text{stepped, cont}} &= (1+T) \cdot C/(D+d_2) \cdot \left[ (\frac{1}{i} - a_S^* - b) - (\frac{1}{i} - a_F^* - b) \right] \\
  z(y)_{\text{FV, stepped}} &= (1+T) \cdot C/(D+d_2) \cdot \left[ (1+i)^{a_F^*} - (1+i)^{a_S^*} - (1+i)^{a_F^*} \right] \\
  &+ (a_F^* - a_F^*) \cdot (1+i)^0 \cdot (y - a_F^* - b)^0 \\
  z(y)_{\text{NPV, stepped}} &= FV_{\text{stepped}} \cdot (1+i)^{a_F^*} - b \cdot (y - 0)^0 
\end{align}

\textbf{Assumptions and Algorithm}

The algorithm of Figure 4 assumes that a complete linear schedule as already been established and is available as the common timeframe in which all elements of cash flows are anchored. For clarity of presenting the new approach and without reducing its correctness, it is further assumed that a single contractor incurs all of the cash flows from the example project, whereas in reality they would be allocated to multiple parties that are connected by contractual relationships. The algorithm is designed to apply singularity function in their logical sequence of transformations from work over time to cost over time. Here a case distinction splits cash outflows from inflows. Interest is applied to the variable balances of the former, which are the discounted to their NPV. The latter growing balances are converted to stepwise billing, increased by a profit factor, decreased by a retainage factor, paid as progress payments, subject to a penalty in case of a schedule overrun, and again discounted to the NPV of the payments, which are then summed.

\textit{<Insert Figure 4 here>
Example

To address Research Objective 4, the example of cash flow for a network schedule of Figure 3 that Elazouni and Metwally (2005) had presented and analyzed originally, is re-analyzed. For consistency with prior studies (Lucko 2011a, Lucko 2011b) it is converted into an equivalent linear schedule as shown on the lower right of Figure 3. Assumptions include the activity costs per month for $A$ to $F$ of $100k, $105k, $110k, $105k, $115k, $105k$, respectively. The interest rate $i_1$ is 0.8% per monthly period and considering variable balances as explained previously; the return rate $i_2$ is omitted in the example but could be implemented analogously; a fee for unused fee from the original example is excluded here because a survey of specialty contractors found that such fee is not used in practice (Lucko 2012). Financing is not constrained by any fixed credit limit, rather, the calculation will yield the maximum balance. All bills will get paid after one period, less a retainage $r$ of 5% that is released as a lump sum with the final payment. The advance payment is deducted again in equal portions from the first five progress payments. Initial costs at startup include $62.1k and $15,962.18 for mobilization and bond, plus an inflow from an advance payment of 10% of the regular project value, which is $161,218.06. Monthly overhead is $32.4k and will accrue until the project finish, which could be extended up to $e = 2$ periods due to $d_1$ and/or $d_2$. Yet a penalty of $2k per period is assessed if the project duration exceeds 5 months; it is deducted from each later payment. A bonus is not specified. Tax is 2% on the activity costs, overhead, and mobilization only. Profit markup is 20% on these three costs as well as tax (Elazouni and Metwally 2005). The following equations model cash flows and NPV.

Once the general work slope of Equation 2 is converted into the cost slope of Equation 9, Equation 14 exemplarily models the cost of activity $A$ where $T$ is the tax rate; others (including overhead) are analogous but omitted here for brevity. Equation 15 is populated with parameter
values for the variable balance of $A$ per the new Equation 8 to give its FV for all possible shifts $d_1$ and rates $d_2$; Equation 16 then discounts its value to time zero. Single cash outflows are the mobilization and bond at time zero; inflows are $161,218.06$ advance at time zero and $1.2 \cdot 0.95 \cdot 63,342$ mobilization (with tax) and $0.95 \cdot 15,962.18$ bond at time two, i.e. the first bill + $b$.

They can be modeled in analogy to single payments as $z(y)_{\text{cost}} = c \cdot (y-a)^6$ where $c$ is negative.

$$z(y)_{\text{cost}}.A = (1+T) - 100k/(D+d_2) \left( (y-a^*_S)^1 - (y-a^*_F)^1 \right)$$ (14)

$$z(y)_{\text{FV cost}}.A = 1.02 - 100k/(1+d_2) \left( 1.008^{1+d_2+\delta} - 1.008^\delta \right) \ln(1.008) \left( (y-a^*_F)^6 \right)$$ (15)

$$z(y)_{\text{NPV cost}}.A = FV_{\text{cost}}.A \cdot (1+i) \left[ a^*_F \right] \cdot (y-0)^6$$ (16)

Equation 17 converts the cost of $A$ into a stepped bill that is paid later ($b$ is one period) in Equation 18, where $P$ is the profit margin and $R$ is the retainage. Equations 19 and 20 convert its payments into a single FV (at $y = \left[ a^*_F \right] + b$) and NPV (at $y = 0$). Five deductions for the advance payment are modeled as $-161,218.06/5 \cdot \left[ 1.008^{-1} - 1.008^{-6} \right] / 0.008$ in analogy to Equation 12.

$$z(y)_{\text{bill}}.A = 1.2 \cdot 1.02 \cdot 100k/(1+d_2) \left( (y-a^*_S)^1 - (y-a^*_F)^1 \right)$$ (17)

$$z(y)_{\text{pmt}}.A = (1-R) \cdot 122.4k/(1+d_2) \left( (y-a^*_S - b)^1 - (y-a^*_F - b)^1 \right)$$ (18)

$$z(y)_{\text{FV pmt}}.A = 0.95 \cdot 122.4k/(1+d_2) \left[ 1.008^{a^*_S+\delta} - 1.008^{a^*_F+\delta} \right] / i + \left( a^*_S - a^*_F \right) \cdot 1.008^{a^*_F+\delta} + \delta^*_A \cdot 1.008^{a^*_F+\delta} \cdot (y-0)^b$$ (19)

$$z(y)_{\text{NPV pmt}}.A = FV_{\text{pmt}}.A \cdot (1+i) \left[ a^*_F \right] \cdot (y-0)^b$$ (20)

Equation 21 is the NPV of a penalty $z(y)_{\text{pen}} = 2k \cdot \left( (y-y_{\text{act}} - b)^1 - (y-y_{\text{act}} - b)^1 \right)$ with the stepped growth of Equation 12, where $y_{\text{act}} = y_{\text{plan}} + e$ is the actual project finish with the delay.

URL: http://mc.manuscriptcentral.com/rcme
e beyond its planned finish at 5 months. An analogous equation could model an early completion bonus by acting before the planned end. The total retainage is fixed at $80,609.03 that is released with the final payment at time $y_{act} + b$ and then discounted by the factor $1.008^{-y_{act} - b} \cdot (y - 0)^0$.

$$z(y)_{NPV_{pen}} = -2k \cdot [1.008^{-y-b} - 1.008^{-y_{act} - b}] / i \cdot (y - 0)^0 \quad (21)$$

The preceding cost and payment equations can be added into one integrated mathematical expression of the entire cash flow profile (Lucko 2011b). However, the focus of this paper lies on moving one step further – collapsing them into the single NPV for financial decision-making.

**Optimization**

*Genetic Algorithm*

Among optimization methods listed by Lucko (2011b) are exhaustive enumeration of all possible solutions, which alas in most projects of real-world size is computationally infeasible; heuristics that apply a set of user-selected empirical rules to determine desirable parameter values; various linear programming approaches, which however only work well for convex solution spaces that are even (meaning there exists a continuous near-proportional behavior between input values and changes in the output); and meta-heuristics. The latter are a family of iterative search algorithms, many of which are inspired by biological phenomena and behaviors. A well-established type of meta-heuristic are genetic algorithms, which are inspired by evolution. Their optimizing process creates sets of potential solutions (chromosomes) that are evaluated (fitness), propagated into the next iteration stochastically based on said fitness (reproduction), and incur spontaneous changes (crossover and mutation) in some of their parameter values (genes) to explore the solution space.

The optimization is computer-implemented modularly; first it performs genetic algorithm steps of initialization, evaluation, reproduction, crossover, and mutation. The evaluation step
calls upon the cash flow module for each solution under consideration. The genetic algorithm is run for 100 iterations only and has four chromosomes of 12 genes each (six shifts $d_1$, six delays $d_2$), which are sequentially randomly sampled with probability $0.5$ for any $d_1$ or $d_2$. Permutations created the set of all possible solutions with a crossover probability of $0.6$ (Lucko 2011a). However, as explained previously, the randomizations within the initialization, crossover, and mutation are all dynamically adjusted within a linear schedule analysis to correctly consider the dependencies within the total project duration (extended if $e$ was non-zero), to always generate only feasible solutions. Any infeasible ones are avoided and the prior version of said chromosome is retained. To rigorously test the iteratively improving findings of this genetic algorithm, its four chromosomes are deliberately seeded poorly (Lucko 2011a) with the known minimum NPV solution in the first iteration, which here was known from a prior enumeration.

**Validation**

Several issues have been identified and solved while verifying the computerization: After several optimization steps the dependencies are checked, requiring sequential calculation of new starts, finishes, and float for each chromosome. It only has to calculate float once, which both $d_1$ and $d_2$ use, despite their different nature. Random sampling in turn is restricted to only within said float to create feasible $d_1$ and $d_2$ values. However, it must be adjusted to be proportional among said feasible solution only, otherwise the infeasible ones would skew the sampling. Experimentation shows how much the reproduction mechanism impacts the performance of the genetic algorithm. Chromosomes are not selected randomly based on percentage differences between NPV values, which are small, but instead labeled ‘4’ to ‘1’ from high to low NPV and selected to reproduce in proportion to these numerical ranks. More sophisticated ranking to magnify differences between
solutions is imaginable, e.g. an exponent of the relative ranks, but is found unnecessary here, the optimization runs successfully with simple ranking. Another issue is that infeasible permutations are numerous and across iterations leads to repeating solutions unnecessarily. The probability of mutation is therefore increased to 1.0 in each chromosome to ensure the search vitality. More observation finds a premature ‘locking’ into clearly suboptimal solutions, as better solutions might differ in more than just one gene. The possible number of mutations per chromosome is thus increased to up to two, which does not complicate the feasibility check. Mutations could increase or decrease the $d_1$ or $d_2$ values. However, a high mutation rate in turn must be balanced with preserving very good solutions, as runs without it are not converging toward an optimum but fluctuate in the long-term. Elitism is implemented in this version of a genetic algorithm in the reproduction step, whereby the solution of highest current NPV is guaranteed to survive into the next generation. It is found that sequential randomization creates a new challenge; later branches (the sequence step II and $d_2$) in the decision tree of conditional probabilities are disadvantaged compared to earlier sampling. However, the rather high mutation probability still facilitates that such permutations are explored, including the eventually found optimum. Modeling both $d_1$ and $d_2$ leads to a complex behavior when genes are varied; it has proven a challenge to explore such uneven solution space with a genetic algorithm and required modifications as described above.

The numerical results of optimizing the NPV for the new cash flow model are validated in two ways: Traditional hand calculations of the NPV for selected permutations of the cash flow profile are compared with a manual evaluation of the new equations for these cases. Then the computer implementation is created, tested for all known cases to verify its correct functioning, and used to perform the optimization. An interesting phenomenon was uncovered: Results for a traditional calculation approach initially did not match the new equations. The former first had to
calculate balances at each integer period, separate the growing partial activity costs in the past period from the single progress payment inflow at that time, and discount them. However, this would not correctly consider that an advance payment ‘lifts’ the entire cash flow profile and thus reduces interest in future periods. A correction would therefore become necessary, to remove such partial positive balances in any periods where a sign change occurs. This correction uses Equation 8 in its form

\[ FV_{bal} = A \cdot \left[ (1+i)^{(1+\delta)} \right] / \ln(1+i) \]

as \( D + \delta = 1 \) for one period. But this extra computational effort is superfluous in the new approach of integrated singularity functions: Here the individual NPV contribution of activities and their eventual payments are fully additive.

The algorithm successfully discards most lower solutions quickly as shown in Figure 5, but in a few cases by pure chance modifies the currently best solution to a slightly lower NPV. The optimum solution of $252,495.58 for the chromosome \{d_{1A}, d_{1B}, d_{2A}, d_{2B}, d_{1C}, d_{1D}, d_{1E}, d_{1F}, d_{2C}, d_{2D}, d_{2E}, d_{2F}\} = \{0, 0, 0, 0, 0, 0, 4, 0, 0\} is known from an exhaustive enumeration, which for this experiment is still computationally feasible. Its duration is 7 days and it requires up to -$517,706.07 of credit. The genetic algorithm has found this optimum in iteration 80 and retains it until iteration 84. However, elitism is not absolute: It is guaranteed in the reproduction step as implemented, but crossover and mutation can modify genes. This allows abandoning the optimum tentatively until finding it again in iteration 98. This performance proves the successful functioning of the optimization, while maintaining its vitality to search the solution space for potentially even better solutions. Table 2 lists various solutions, e.g. the baseline when all \( d_1 \) and \( d_2 \) are zero. Boldface marks maxima and minima. These models were optimized by either NPV and profit as their criterion as indicated. As evidence of the superiority of the new approach to cash flow modeling that considers the Time Value of Money, several comparisons are made: If optimization would be merely based on profit, the solution of $269,163.32 would be selected.
However, this has an NPV of $251,869.57, which is $626.01 lower than the maximum possible NPV of $252,495.58 (although conversely having a lower profit). This ‘incorrect’ financial decision-making using profit underlines the importance of optimizing by NPV so that the best value for the project will indeed be achieved. Interestingly, the minimum solutions are the same at $244,848.20 (NPV) and $256,634.85 (profit). The critical contribution of the new approach particularly comes to bear for the maximum value case. Several non-optimal solutions (all for 7-month-long projects) are also listed. They illustrate the counterintuitive relationship of NPV and profit: Comparing solutions, for example $252,451.87 and $252,295.35 (both NPV), the former has the higher NPV but ‘disproportionately’ a lower profit ($267,648.82 versus $268,106.02), but for a pair of solutions with a larger difference, e.g. $246,008.77 and $244,848.20 (NPV) their profits conversely have a ‘proportional’ relationship of $257,572.39 versus $256,634.85 (profit). In other words, NPV and profit are in a non-trivial and complex relationship with each other.

In this example (Elazouni and Metwally 2005) the NPV of the entire project reaches its maximum when the relatively inexpensive single activity C is shifted to the project finish that is extended by e, to keep earning income from overhead costs. Further high NPV solutions in Table 2 confirm that extending D, E, and F has a similar effect on cash flows, as do others, e.g. C and D combined. In the given inputs the overhead exceeds the penalty, which could entice extending the project improperly to exploit its cash flows. Owners in reality would obviously consider this undesirable. Specifically, a 7-month-long project has an NPV = $252,495.58, while a 5-month-long one only yields an NPV = $244,848.20, or $7,647.38 less. This phenomenon prevents an otherwise interesting comparison of short versus extended projects in Table 2, which could have given insights into the tradeoff in terms of their duration versus NPV. Another consideration is that deliberately delaying activities increases their risk, as they consume float and are left with
less flexibility to mitigate sudden external changes. Quantifying such risk and its potential costs, and comparing them with improvements in the cash flow is beyond the scope of this research.

Conclusions

A new approach to model, analyze, and optimize cash flows in construction management based on singularity functions has been presented. It improves upon previous studies by focusing on the net present value rather than merely maximizing profit. It also links all transactions with their time aspect, here in form of a linear schedule. The new model seamlessly connects all cash flow elements: Costs that can occur pointwise or distributed are aggregated into regular bills, which eventually become payments, subject to terms and conditions. Any periods of negative balances will cause financing fees to the contractor, which are assessed and charged at the ends of periods. Unlike previous studies, the cash flow profile does not consist of disjointed discrete amounts, but rather is captured in its entirety as a single mathematical expression. Its variables for amount and timing of cash outflows and inflows provide a unique property that facilitates optimization: One function can model all possible permutations of a cash flow profile and their respective NPV.

Contributions to the body of knowledge are fourfold per the stated Research Objectives:

An ability to freely convert network schedules into higher-dimensional and more information-rich linear schedules, for which costs and payments can then be analyzed; an exact derivation of interest on partial variable balances; a cash flow analysis algorithm that explicitly considers the
Time Value of Money in form of the net present value for decision-making; and an optimization approach that obeys the dependencies between activities, whose performance has been validated.

Assumptions of the approach include that it simplified reality by treating activities and events that impact cash flows as known or quantifiable in advance, omitting variability besides shifts and delays, and considering only a single contractor, whereas a real construction project is typically performed by many subcontractors that are hired by the general contractor. In practice the cash flow profile of a complete project is a composite of those of its multiple participants, who may be subject to different financial terms and conditions. Nonetheless, the model remains generally applicable, while the quality of its results depends on the level of detail and realism of its inputs. Another simplifying assumption is the use of a single interest rate for discounting and compounding in the model, whereas companies typically have different types of sources of funds at their disposal in practice. Depending on the company type and capabilities, these can include company capital, loans, bonds, and shares from which the overall budget could be assembled. Finally, it is an assumption that all costs have been derived from and directly tied to underlying activities in the linear schedule. Individual transactions could occur long before or after their respective construction operation is performed. To preserve the general nature of the approach, ‘activity’ therefore should be taken to mean any administrative or productive task that provides a relative time reference for the transaction. As modeled, it can either be fixed at a given date (e.g. mobilization), variable within a range of time (e.g. activity within its float), or even conditionally depend upon specific points in time, cash balances, or other conditions (e.g. bonus or penalty). Such different types of cash flows can all be expressed as singularity functions. A challenge has been encountered in incorporating the dependencies within schedules. They require a sequential sampling procedure that permits to use only feasible values in each step. Shifts and delays cause
interactions between activities that create complexity in form of an uneven solution space, which is somewhat difficult to explore with optimization methods. However, the genetic algorithm has exhibited a converging behavior and has proven to successfully maximize the net present value.

The performance of the genetic algorithm has been found to be sensitive to its parameter settings. Further research on how to robustly explore such search spaces therefore is merited. An observation on the particular example as obtained from the literature is that its penalty is too low to offset a possible profit from extra overhead. More realistic assumptions and inputs should thus be used in future analytical calculations, ideally through case studies of representative industry projects. On the operations side, singularity functions could connect cash flows and schedules with inventory management, inspired by their use in resource management. On the financial side, they could capture more terms and conditions, e.g. cash or trade discounts that incentivize larger orders or faster payments, e.g. for suppliers or subcontractors, raising the question at which values to offer or accept them. This could be expanded toward multi-objective optimization to minimize schedule duration and improve resource use while maximizing overall financial value.

Moreover, variability could be incorporated in values of $d_1$ and $d_2$ for different types of activities. This would allow a probabilistic analysis to provide decision-makers with quantitative guidance on the likelihood of incurring certain cash flow scenarios and developing strategies to improve net present values. Given the uncertainty in construction projects, a connection with risk management exists insofar as any financial benefits in net present value from delaying activities and events should be contrasted with the thereby increased risk. Further research thus is needed to compare the implications of different cash flow management strategies, for example front-loading billable expenses or unbalancing the profit margin across time. Issues that arise include the impact of considering multiple interest and/or investment rates (as firms realistically do not
rely on only one source but have mixed financial portfolios); seeking to find an optimal mix of
debt and equity financing; exploring the impact of using the weighted average cost of capital
(Miles and Ezzell 1980); and moving from the current focus on net present value toward an even
more comprehensive framework of maximizing overall shareholder value (Lazonick and O’Brien
2000) for the case of publicly traded firms, within which cash flow is merely one component.
Such broad model would require considering many issues that add significant complexity and
exceed the scope of this discussion, e.g. the composition of debt versus equity of a company that
influence its creditworthiness and cost of capital, the objective valuation of capital projects by
investors, and the effectiveness of markets in which various financial instruments are traded.

Time Value of Money plays a critical role in financial analysis, because it allows directly
comparing the value of different options to execute a project in its planning and budgeting phase,
and can also be useful for controlling and managing yet-to-be-built portions of a project in terms
of their contribution to the overall financial performance. Improving analytical capabilities for
cash flows to this effect can help contractors compete in their challenging market environment.
Accordingly, the new approach primarily gains value by its ability to support financial decision-
making in maximizing the net present value within the opportunities that the project constraints
allow. Besides opening a new avenue to the Time Value of Money, singularity functions can also
inspire further theory-building research to model financial phenomena in an integrated manner.
References


URL: http://mc.manuscriptcentral.com/rcme


URL: http://mc.manuscriptcentral.com/rcme


Figure 1: Superposition of basic terms for singularity function
Figure 2: Variable balance across multiple periods
Figure 3: Network, bar chart, and linear schedule of example with potential extensions (after Elazouni and Metwally 2005)
**Step 1: Previous Schedule Analysis**
Six previous steps: Capture schedule data, initial activity and buffer equations, differences thereof, differentiate differences, final activity equations

**Step 2: Establish Cash Flow Model**
Insert unit costs for cost equations, convert into stepped bills, use profit factor, shift until payment, apply retainage factor, add to cash flow equation

**Step 3: Finance Variable Balance**
Distinguish single, growing, and stepped flows, apply new interest term on variable balance, add penalty or bonus terms and retainage if applicable

**Step 4: Singularity Functions**
Establish singularity function for each direct and indirect cost element depending on its time-dependent behavior, add, sort, and simplify terms

**Step 5: Decision Measure at Common Time**
Consider Time Value of Money by discounting toward origin, insert shift and rate variable values as necessary, optimization toward maximum NPV

---

*Figure 4: Flowchart of Analysis Steps*
Figure 5: Genetic algorithm performance
Table 1: Measures of Financial Performance for Decision-Making

<table>
<thead>
<tr>
<th>Name</th>
<th>TVM</th>
<th>Equation</th>
<th>Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit</td>
<td>No</td>
<td>Revenues – expenses</td>
<td>N/A</td>
</tr>
<tr>
<td>Profitability Index</td>
<td>Yes</td>
<td>Σ PV of all CF / initial investment</td>
<td>Traditional CF only, discount rate</td>
</tr>
<tr>
<td>Payback Period</td>
<td>No</td>
<td>Initial investment / average period inflow</td>
<td>Traditional CF only</td>
</tr>
<tr>
<td>Discounted Payback Period</td>
<td>Yes</td>
<td>Same, but Σ discounted CF</td>
<td>Traditional CF only</td>
</tr>
<tr>
<td>Net Present Value</td>
<td>Yes</td>
<td>Σ PV of all future CF – initial investment</td>
<td>Discount rate</td>
</tr>
<tr>
<td>Net Uniform Series</td>
<td>Yes</td>
<td>Annuity equal to NPV</td>
<td>Discount rate</td>
</tr>
<tr>
<td>Internal Rate of Return</td>
<td>Yes</td>
<td>Discount rate at zero NPV</td>
<td>Traditional CF only, hurdle rate</td>
</tr>
<tr>
<td>Modified Internal Rate of Return</td>
<td>Yes</td>
<td>( n )-th root of FV of inflows / – PV of outflows</td>
<td>Discount and compound rates</td>
</tr>
<tr>
<td>Accounting Rate of Return</td>
<td>No</td>
<td>Average profit / average book value</td>
<td>Profit, depreciation</td>
</tr>
</tbody>
</table>

Note: TVM is Time Value of Money, PV and FV are present and future value, CF are cash flows.
### Table 2: Comparison of Selected Solutions for Linear Schedule Cash Flow Example

<table>
<thead>
<tr>
<th>Remarks</th>
<th>NPV</th>
<th>Profit</th>
<th>d_{1A} d_{1B} d_{2A} d_{2B} d_{1C} d_{1D} d_{1E} d_{2C} d_{2D} d_{2E} d_{2F} Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. NPV</td>
<td>$252,495.58</td>
<td>$267,876.20</td>
<td>0 0 0 0 0 0 0 4 0 0 0 7</td>
</tr>
<tr>
<td>Min. NPV</td>
<td>$244,848.20</td>
<td>$256,634.85</td>
<td>2 0 0 0 1 1 0 0 0 0 0 5</td>
</tr>
<tr>
<td>Baseline</td>
<td>$246,008.77</td>
<td>$257,572.39</td>
<td>0 0 0 0 0 0 0 0 0 0 0 5</td>
</tr>
<tr>
<td>Max. Profit</td>
<td>$251,869.57</td>
<td>$269,163.32</td>
<td>0 0 0 0 0 0 2 0 1 0 0 1 7</td>
</tr>
<tr>
<td>Min. Profit</td>
<td>$244,848.20</td>
<td>$256,634.85</td>
<td>2 0 0 0 1 1 0 0 0 0 0 5</td>
</tr>
<tr>
<td>High NPV</td>
<td>$252,451.87</td>
<td>$267,648.82</td>
<td>0 0 0 0 0 0 0 0 0 0 0 2 7</td>
</tr>
<tr>
<td>High NPV</td>
<td>$252,372.96</td>
<td>$268,035.98</td>
<td>0 0 0 0 0 0 0 0 0 0 0 0 7</td>
</tr>
<tr>
<td>High NPV</td>
<td>$252,295.35</td>
<td>$268,106.02</td>
<td>0 0 0 0 0 0 0 0 0 0 0 4 7</td>
</tr>
<tr>
<td>Other NPV</td>
<td>$252,396.68</td>
<td>$267,650.99</td>
<td>0 0 0 0 0 0 0 0 0 1 5 0 0 7</td>
</tr>
<tr>
<td>Other NPV</td>
<td>$252,389.37</td>
<td>$267,765.71</td>
<td>0 0 0 0 0 0 1 0 0 4 0 0 7</td>
</tr>
</tbody>
</table>
### Notation

#### Symbols
- \( a \) = range cutoff of singularity function
- \( b \) = billing-to-payment delay
- \( d \) = variable time component (shift or delay)
- \( C \) = total cost of activity
- \( c \) = increment of cost
- \( D \) = planned activity duration
- \( FV \) = future value
- \( h \) = substitution for \( 1/k \)
- \( i \) = interest rate
- \( k \) = number of infinitesimally short periods
- \( NPV \) = net present value
- \( n \) = behavior exponent of singularity function
- \( P \) = profit rate
- \( PV \) = present value
- \( p \) = number of periods
- \( R \) = retainage rate
- \( s \) = strength factor of singularity function
- \( T \) = tax rate
- \( t \) = increment of time (duration)
- \( u \) = generic unit of work amount
- \( W \) = activity work amount
- \( x \) = dependent variable along vertical axis, here work amount
- \( y \) = independent variable along horizontal axis, here time
- \( z \) = dependent variable along vertical axis, here cost
- \( \langle \rangle \) = brackets of singularity function
- \( \delta \) = partial period from activity finish to next integer time
- \( v \) = substitution for \( 1 + i \)

#### Subscripts
- \( act \) = actual project finish
- \( ann \) = annuity
- \( bal \) = variable balance
- \( F \) = finish of activity segment
- \( i \) = counting index for activity segments
- \( inflows \) = index for cash inflows
- \( NPV \) = net present value
- \( outflows \) = index for cash outflows
- \( pen \) = penalty
- \( pmt \) = payment
- \( plan \) = planned project finish
- \( S \) = start of activity segment
- \( stepped \) = incremental repeated cash flow with stepped cumulative value
- \( time \) = time over work equation (inverse proportional to productivity)
- \( work \) = work over time equation (proportional to productivity)
1 = index for time shift
2 = index for time delay

Superscripts
* = variable that includes shifts and/or delays