
Introduction
The author is commended for a very insightful experimental study, the likes of which are seen only all too few in construction engineering and management publications. This discussion is intended to augment a few minor items in the paper, which can help lending support to its interesting premise, that an ingenious application of geometric principles could have made vast practical contributions to an enormous engineering undertaking.

Curve Shape and Equation
One of the two major tools described in the paper is the ‘rocker’ that was hypothesized to have been used to move blocks. It was alternatively but somewhat inaccurately called a quarter circle (Lepre 1990). Circle segments are cutoffs from the edge of a full circle, whereas quarter circles are pie-shaped wedges and can thus have a higher material use. Considering the still unknown exact shape, this tool will herein be referred to as a rocker. Two hypotheses were presented (Fonte 2011) as to the geometric shape of this rocker: A circle segment or an inverted catenary curve, whose modern name means chain-shape, in reference to its derivation as the shape of a rope or chain of some mass that is freely suspended between two anchorages and sags in the middle due to gravity. The equation for the height of a circle segment (Stöcker 1993 p. 82f.) is $h = r \cdot (1 - \cos(\varphi/2))$ and for its width $w$ is $w = 2 \cdot r \cdot \sin(\varphi/2)$, where $\varphi$ is the opening angle of the segment relative to the circle center, and $r$ is the radius, which in a unit circles is set equal to one. The equation for an inverted catenary curve in $x$-$y$-coordinates is $y(x) = a \cdot \cosh(x/a)$, where the...
shape parameter $a$ describes width and deflection (Stöcker 1993 p. 287). For a rope of length $l$
over a span $w$ the sag $h$ is approximately (Gross et al. 1995, p. 156) $h=\sqrt{(3/8)\cdot(l-w)\cdot w}$. Setting
the width $w$ and variable zenith heights $h$ of both curve types equal in said unit circle allows
comparing their shape as shown in Figure 1. Percentage differences between circle segments and
equivalent catenary curves for different ratios $h/r$ per Table 1 are in fact only small and deviate
from it at a similar magnitude as another well-known shape, a parabola. Catenary curves at
different values of $w$ are consistently lower than equivalent circle segments, more pronouncedly
so when approaching a semicircle at smaller radii.

Insert Figure 1 here

Insert Table 1 here

An alternative hypothesis for the rocker was presented by Fitchen (1978), who described how a
single rocker may have been used as a stationary vertical lifting apparatus, where rocking a block
to one side alternated with inserting a thin layer of wedge risers on the other side and vice versa,
so that the block was jacked upward. The discussion of how blocks were moved – using sleds as
shown on frescos on a ramp of controversial layout (straight, segmental, or spiraled), perhaps
aided by rollers, counterweighted levers with inspiration from water bucket lifts, or other
mechanisms continues unabatedly (Edwards 2003). For the hypothesized movement of a block
with square cross-section that is rolled over a track of rockers so that its centroid always remains
at the same height over ground to minimize the energy use (Fonte 2011), the exact mathematical
solution is a catenary curve, as mentioned: “It was subsequently learned that this shape is a
catenary, and that rolling a square over this shape had been noted previously” (Fonte 2011, p.
1197). The seeming intricacy of modeling a catenary curve, which was first mathematically
explored by Robert Hooke in 1675 (Osserman 2010) appear to have led to the conclusion that it
is somehow more complex to conceive or execute than a circle segment: “It is not suggested that the Egyptians performed the complicated mathematics needed to create the quarter-circles from theory. Rather, they probably started with a circular shape, which is only in error by a small percent. They could have then refined this shape through observation” (Fonte 2011, p. 1198).

Here a widespread but incorrect assumption arises, to which many other authors succumbed (Herz-Fischler 2000), that an ability to perform mathematics analytically is required to practically construct the shape that it describes successfully.

Mathematical versus Practical Knowledge

Another unsolved question from ancient Egypt illustrates this point, whether Egyptians knew the value of $\pi$ (Pi) = 3.1415... and somehow deliberately encoded it, or even other universal constants (Boorstin 1992), into the pyramids: The side angle of most Egyptian pyramids is approximately 51°52’ (51 degrees, 52 minutes of arc) (von Ditfurth and Arzt 1982), an uneven value in degrees compared to other possible triangular shapes that could be derived from 30°, 45°, or 60° slopes. The original dimensions of the Great Pyramid of Khufu were about 148 m (485 ft) tall and 233 m (763 ft) wide (Fonte 2011), or 146.6 m (481 ft) and 230.4 m (756 ft), according to different sources (Jackson and Stamp 2003). Its angle is close to the ratio of its circumference divided by twice its height, which led von Ditfurth and Arzt (1982) to hypothesize that the Egyptians used a simple geometric approach to derive the universally known pyramid shape: Roll off one circumference of a circle with unit radius exactly once, and then stack the same circle four times above each other at the end point, which yields an angle of $2 \cdot \pi / 8 = 1.2732... = 51.8539... \approx 51°51’$. Three pyramids that were built under Sneferu may support this intriguing hypothesis, here listed in chronological order of their construction: The ancient structural failure of the Maidum Pyramid (Mendelssohn 1973), an unusual Bent Pyramid, whose
angle was reduced during construction, and the nearby Red Pyramid, both with a lower angle of
43.5° (Mendelssohn 1971) or 43°22’ (Jackson and Stamp 2000). These angles are near (but not
identical to) the value from stacking the circle only three times (von Ditfurth and Arzt 1982), i.e.
2 \cdot \pi / 6 = 1.0471... = 46.3207...° \approx 46°19’. It is possible that such slight discrepancies in the
g eo metrically derived proportions stem from rounding to even values in Egyptian units, just as
carpenters use ratios of small integers for gradients. One royal cubit of seven palms was about
52.3 cm (20 \text{"}"/16 in), Scott 1942). For the Great Pyramid the dimensions of 280 cubits tall and
440 cubits wide (Petrie 1883) gives 51.84277341...° and could lead to the unsupported notion
that \pi was encoded as (4 \cdot 440) / (2 \cdot 280) = 22/7 = 3.1428.... The Red Pyramid was said to have
been designed as 200 cubits high and 420 cubits wide (Miatello 2005), i.e. at an angle of
43.6028...°. Yet as pyramids over millennia have lost almost all of their external façade of
polished casing stones, it appears notoriously difficult to survey and measure them accurately.
Geometric evidence should therefore be combined with archaeological findings (Krauss 1996).

However, none of these interesting considerations of pyramid proportions require any knowledge
of the exact value of \pi. In fact, tracing a catenary under field conditions is just as easy as a circle
segment; both require a rope and a pen or stylus to trace the outline. The only difference is that a
circle segment would trace an arc with the rope as its radius, on a horizontal surface, whereas a
catenary shape could be traced by suspending the rope next to a vertical plane or board. Rockers
of either type could thus have been built from experimentation without any mathematics, just as
the pyramids proportions simply reflect the application of geometry that the ancient Egyptians
used for designing them. From this point of view both a catenary curve or a circle segment would
be acceptable explanations.
It is noted that if the dimensions of a rocker could be measured or reconstructed very accurately in its current state of preservation, it might be possible to hypothesize further about its actual shape and use: If the rocker indeed was a perfect circle segment as the author assumes, it could have been used to transform a block of square cross-section into a wheel shape by wrapping rockers around it on all four sides with ropes (rectangular blocks would have required two different types of circle segments). If, however, the rocker was a catenary curve, it would be more likely that it was indeed used as a track. This has two advantages: First, it reduces the number of rockers by half to only two per block (one that the block has just left and one onto which is has just moved, so that workers could lift and replace rockers akin to modern track-type heavy construction equipment). Second, it will distribute the load of the block as pressure across the entire footprint of the rocker (Fonte 2011), akin to a track-type vehicle. This is more beneficial as it prevents penetrating into the soil when moving the block, instead of turning a block into a ‘wheel’ that may still become stuck. However, the very small differences per Table 1 suggest that it may be impossible to solve the question of the rocker shape completely.

Related Examples of Curve Usage

Curved shapes occur in nature and are used in early construction. The former may have inspired led to the latter, for example hanging vines may have developed into suspension bridges. Shipbuilders since antiquity have successfully faced the challenge of creating a three-dimensionally curved shape. They developed splines, i.e. thin bendable strips, to draw energy-minimizing curves between two or more fixed nodes, which only in the 20th century were translated into mathematical theory (Schoenberg 1946), again showing that practice by using tools can precede theory with mathematical analysis. An abstraction, however, opens avenues to
further beneficial applications that may not be apparent from the original context, e.g. the modern use of spline functions for scalable computer fonts.

Many ancient arch structures use circle segments, as is most well known from the Roman era. Even later pointed arches of the Gothic period were typically derived geometrically using compass and ropes (Toker 1985), which is described in documents by their actual builders. Only then can the original design intent and construction techniques be known with certainty, just as they are for what is arguably the most well-known structure with a catenary shape, the Gateway Arch in St. Louis. It was designed as a modified inverted catenary shape on an inverted triangular footprint, tapering toward its apex (Moore 2005).

Catenaries have been applied in experimental archaeology and educational contexts. For example, Peterson (2004) described a square-wheeled tricycle that can ride on a track of catenary segments, just like the rockers described by Fonte (2011). Wallington (2003) provided descriptions and video clips of how he moved concrete blocks of dimensions as had been used in Stonehenge using levers and counterweights, and specifically rolled a block across a track of curved wooden segments, whose shape he created with a spline. Similar experiments on moving heavy blocks were reported by Pipes (2003) and various other authors, who have broadened the question from pyramid blocks and also obelisks in ancient Egypt to other cultures and time periods, including, but not limited to, Megalithic sites in Europe, Ancient Greece and Rome, China, Meso-America, and Easter Island. In conclusion, each of these studies, including Fonte’s (2011) detailed experiments, can add evidence toward understanding how seemingly impossible engineering challenges may have been accomplished by ancient builders, whose structures prove that it was possible.
Acknowledgement

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References


Table 1: Percent Difference of Height for Catenary Curves versus Circle Segments

<table>
<thead>
<tr>
<th>Height Ratio ( h / r )</th>
<th>At ( w = 0.25 )</th>
<th>At ( w = 0.50 )</th>
<th>At ( w = 0.75 )</th>
<th>At ( w = 0.95 )</th>
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</thead>
<tbody>
<tr>
<td>1.00</td>
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<td>-26.37%</td>
<td>-62.94%</td>
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<td>-6.62%</td>
<td>-18.17%</td>
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<td>-3.56%</td>
<td>-9.04%</td>
<td>-16.87%</td>
</tr>
<tr>
<td>0.25</td>
<td>-0.25%</td>
<td>-1.01%</td>
<td>-2.35%</td>
<td>-3.91%</td>
</tr>
</tbody>
</table>
Figure 1: Comparison of Circle Segments and Equivalent Catenary Curves