Modeling Cash Flow Profiles with Singularity Functions

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Abstract

The ability to plan and manage cash flows is critical for the survival and long-term economic success of construction contractors. Profiles arise from the interplay of outflows, e.g. labor, equipment, and materials, and inflows, e.g. progress payments less retainage, which vary in frequency and are subject to payment terms. Their complex zigzag shape was previously only modeled with simplifications. Approaches included discrete values at specific points in time; averaged S-curves, e.g. from regression; or envelopes of extreme values. Yet neither is suited for cumulative costs of variable schedule activities. This paper therefore describes how cash flow profiles can be flexibly and accurately modeled with singularity functions, which originated in structural engineering. Their components define ranges of behavior between cutoff values. Emphasis is placed on expressing common payment terms, which are illustrated with an example that is validated with the literature. This new approach augments project planning toward an integrated model.

Introduction

The success of construction contractors relies upon sound financial management of their portfolio of projects. While profits or losses materialize in the balance sheet only at completion of a project, cash is a tangible and depletable (yet reversible) resource whose scarcity can cause disruptions or even the eventual bankruptcy of a firm. Cash is thus considered the most important element within construction projects (Hwee and Tiong 2001). The complex behavior of cash flows arises from an interplay of many transactions that depend on time, events, or prior costs. Forecasting and managing them therefore requires proper models and tools. Building upon previous work on analyzing linear schedules (Lucko 2009), this paper presents a new approach to accurately represent cash flows while also facilitating an intuitive understanding.

Importance of Cash Flows

Careful “cash flow management is critical… to the heart of the financial viability of a construction company” (Kenley 2003, p. 162). “The prudent contractor therefore... takes the cash flow issue very seriously” (Touran et al. 2004, p. 719), as obtaining a
bank loan can be difficult for a risky business like construction. Capital enables the vital debt-to-income ratio, which is indispensable to achieve profitability (Chen et al. 2005). Its lack may even lead to eventual bankruptcy (Harris and McCaffer 2001).

**Previous Limitations**
Previous studies faced the challenge of the “'sawtooth’ effect [that] is very difficult to model. It is quite suitable for representing real data, but for any form of modelling, forecasting or simulation it cannot be represented” (Kenley 2003, p. 168). Oscillating cash inflows and outflows thus were simplified as discrete values (Khosrowshahi and Kaka 2007), empirical averaged S-curves (Park et al. 2005) without peaks, which was deemed inaccurate (Kaka and Evans 1998), or envelopes (Hwee and Tiong 2001).

“Due to a lack of accurate prediction models... [Texas DOT] usually estimates payments as uniform distributions over the project duration” (Jarrah et al. 2007, p. 235). Complexity also stems from variable cash flows occurring at either the activity, project/program, or company levels (Navon 1996) and complicates optimizing them.

**Analytical Approach**
Singularity functions offer an elegant way to model and analyze complex phenomena. Used in structural analysis, they alleviated the need to solve many separate equations and boundary conditions and instead expressed the entire scenario with one equation of connected ranges that start wherever any variable of interest changes its behavior.

**Definition of Singularity Functions**
Singularity functions are range-based and contain at least one basic term of Eq. 1. Its pointed bracket operator is a case distinction that ‘activates’ it only from \( a \) onward.

\[
y(x)=s \cdot \delta(x-a)^n = \begin{cases} 
0 & \text{for } x < a \\
q & \text{for } x \geq a
\end{cases}
\]

Equation 1

where \( x \) is the independent variable on the horizontal axis, \( y \) is the dependent variable on the vertical axis, \( a \) is the cutoff \( x \)-value, \( n \) is the order of the curve segment that is modeled, and \( s \) is a scaling factor. Thus e.g. \( n = 0 \) is a step with constant height \( s \) at \( a \) and \( n = 1 \) a ramp with slope \( s \) from \( a \). For example, a linear equation \( y(x) = m \cdot x + b \) of slope \( m \) from intercept \( b \) gives the singularity function \( y(x) = b \cdot <x - 0>^0 + m \cdot <x - 0>^1 \) with two terms that separately model the initial intercept and subsequent slope.

**Superposition Principle**
It is possible to describe any complex shape by additive superposition in analogy to structural analysis that modeled e.g. several point and distributed loads along a beam. An unlimited number of basic terms of Eq. 1 can be included in one singularity function. Modeling such complex shape divides its geometry into a group of range segments of basic behaviors. For illustration, the low-complexity shape of Figure 1 is modeled explicitly by Eq. 2. Any higher order terms could be included analogously.

\[
y(x)=1 \cdot <x-0>^0 + \frac{1}{1} \cdot <x-0>^1 - \frac{1}{1} \cdot <x-1>^1 - 1 \cdot <x-2>^0 - \frac{1}{2} \cdot <x-2>^1
\]

Equation 2
The edge of a shape, shown as a thick solid line, is composed by adding several areas whose borders are shown as dashed thin lines. The signs in Figure 1 indicate positive and negative areas, i.e. added upward or downward on the y-axis. Horizontal stripe I is added first, then triangle II. Triangle III, horizontal stripe IV, and triangle V are subtracted. These five areas match with the five basic terms of Eq. 2. The height in the y-direction is always correct, even if basic terms with the same $a$ are switched, e.g. if triangle II were added before stripe I. The dashed rightmost edge indicates that from their cutoffs $a$ onward all basic terms remain active until infinity on the $x$-axis.

**Simplification of Terms and Sorting**

Eq. 1 acts like an off-on switch and singularity functions are evaluated *cumulatively*. The further one moves into the positive $x$-direction the more terms become active. Note that basic terms contribute *changes* to the behavior of the overall singularity function. Several basic terms $i$ can be *simplified* using Eq. 3 into one term by adding their $s_i$ if – and only if – they have identical values of their cutoff $a$ and exponent $n$.

$$s_1 \cdot (x-a)^n + s_2 \cdot (x-a)^n = (s_1 + s_2) \cdot (x-a)^n$$

Eq. 3

Despite their name, singularity functions are right continuous and defined at all $x$-values (or fall to zero). They can be differentiated and integrated normally (Lucko 2009). For clarity and ease of the manual calculations, it is much recommended that all basic terms are sorted from left to right by their $x$-value and within that by their $n$.

**Cash Outflows**

Expenses, or cash outflows from the viewpoint of a contractor, are incurred for every line item, e.g. to pay labor, rent equipment, purchase materials, and cover overhead of the contracting firm. As in most models (Elazouni and Metwally 2005), costs in this paper are assumed to be simplified as growing linearly for each activity individually, but could also consist of many small transactions. Such growing cash outflows can be modeled as one singularity function per activity, plus one function that describes any non-activity costs, e.g. overhead. The rate at which money is spent over time becomes
their ‘cost slope.’ For a more detailed model the individual costs can be modeled with Eq. 2. Each expense would be one basic term of Eq. 1. Both such models, linear outflows or the detailed stepped outflows, require that a final change term is included, which deducts all previously accumulated costs again if fluctuations over time are of interest (in analogy to a density function) instead of only the cumulative total cost. Additional cash outflows occur from financing interest that is assessed by the lender in regular intervals of integer time units, e.g. months and is applied directly after the end of each period. A new variable $\varepsilon$ is introduced to correctly model this minute increment of time, which is set to an infinitesimally short duration. Otherwise, the singularity function for interest would incur a logical impossibility of having to evaluate the overdraft at the same point in time at which the interest itself is added.

Cash Inflows

Contractors add profit on their bills to the owner. Assuming that profit is applied evenly to all cash outflows, they are multiplied by a factor of $1 + p$, where $p$ is the profit percentage. Otherwise, each type of costs could have a different value for $p$. Bills, i.e. requests for progress payments, extend a payment period to the owner that ends with the due date. Unless early payment discounts are offered, owners will often seek to pay bills as late as possible to continue earning interest on their cash holdings. Such billing-to-payment-delay $b$ is modeled by adding $b$ to each billing date in the singularity function. Contractors must bear expenses for extended durations that can reach 90 days (Setzer 2009) and require financing. Such delays result from ‘pay when paid’ (Blyth and Kaka 1999) provisions. They especially affect subcontractors, who accumulate their expenses into a bill to the general contractor, who bills the owner, who in turn sends the payment less a retainage to the general contractor at the end of the contractually permissible payment period, and it is finally passed on downward.

Payments, or cash inflows from the viewpoint of a contractor, are received at a steady frequency whereas outflows grow continuously. Expenses must be aggregated into an individual regular payments that are (a) delayed by $b$, (b) reduced by the retainage if such is withheld, and (c) are stepped at each period. These modeling objectives can be accomplished with floor and ceiling functions. They were defined by Iverson (1962) for a programming language that used the operators $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ to round a real number downward or upward to integers. For modeling cash inflows, $\lfloor \cdot \rfloor$ is applied to the time variable of the singularity function. Deducting them from the cumulative total of cash inflows and outflows ensures that the correct expenses are billed after each period.

A retainage $r$ of 5% to 10% of the bill is typically withheld by the owner to provide an incentive for the contractor to finish the work. It is released only after substantial completion (Touran et al. 2004), which burdens subcontractors beyond their own portion of the work. It is modeled by setting up two singularity functions, one of $1 - r$ while retainage is withheld and another one of $r$ to give the retained amount itself.

Cash Flow Example

The example of Figure 2, analyzed by Halpin and Woodhead (1998) in tabular form, illustrates how to model cash flows. Table 1 summarizes the activity dates and their unit costs. It assumes $p = 25\%$ profit on all costs and $r = 10\%$ retainage until the project exceeds $125,000 in value and is released with the final payment. Overhead is
$5,000 monthly (dashed line). The billing delay $b$ is one month, shown as arrows. It causes a lag in Figure 2 between the outflow (without and with profit, upper solid and dotted lines) and stepped inflow curve, whose difference is also shown below the time axis. The vertical $z$-axis for cost and a horizontal $y$-axis for time are consistent with linear schedule analysis (Lucko 2009). The interest $i$ is “1% per month for the amount of the overdraft at the end of the month” (Halpin and Woodhead 1998, p. 123). The model for the costs of each activity plus overhead creates five equations. Only Eq. 4 of activity $A$ is shown for brevity. Its costs are added from its start and then subtracted again at its finish; otherwise they would continue to accrue until infinity on the $y$-axis.

![Figure 2: Cash Flow Example (Halpin and Woodhead 1998)](image)

**Table 1: Activity List for Cash Flow Example (Halpin and Woodhead 1998)**

<table>
<thead>
<tr>
<th>Activity</th>
<th>Duration</th>
<th>Start</th>
<th>Finish</th>
<th>Total Cost</th>
<th>Unit Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2.0 mo.</td>
<td>0.0 mo.</td>
<td>2.0 mo.</td>
<td>$50,000</td>
<td>$25,000/mo.</td>
</tr>
<tr>
<td>B</td>
<td>2.0 mo.</td>
<td>1.0 mo.</td>
<td>3.0 mo.</td>
<td>$40,000</td>
<td>$20,000/mo.</td>
</tr>
<tr>
<td>C</td>
<td>1.5 mo.</td>
<td>1.5 mo.</td>
<td>3.0 mo.</td>
<td>$60,000</td>
<td>$40,000/mo.</td>
</tr>
<tr>
<td>D</td>
<td>2.0 mo.</td>
<td>2.0 mo.</td>
<td>4.0 mo.</td>
<td>$30,000</td>
<td>$15,000/mo.</td>
</tr>
<tr>
<td>Overhead</td>
<td>4.0 mo.</td>
<td>0.0 mo.</td>
<td>4.0 mo.</td>
<td>$20,000</td>
<td>$5,000/mo.</td>
</tr>
</tbody>
</table>
\[
\begin{align*}
\text{Eq. 4} & \quad z(y)_\text{t} = 0 \cdot (y-0\text{mo.})^0 + \frac{25,000}{1\text{mo.}} \cdot (y-0\text{mo.})^1 - \frac{25,000}{1\text{mo.}} \cdot (y-2\text{mo.})^1 \\
\text{The total costs without profit are given by the superposed and simplified Eq. 5, which} & \quad \text{is the upper solid line in Figure 2. Multiplying it by 1.25 gives the costs with profit.} \\
\text{Eq. 5} & \quad z(y)_{\text{cost+OH}} = 0 \cdot (y-0\text{mo.})^0 + \frac{30,000}{1\text{mo.}} \cdot (y-0\text{mo.})^1 + \frac{20,000}{1\text{mo.}} \cdot (y-1\text{mo.})^1 + \frac{40,000}{1\text{mo.}} \cdot (y-1.5\text{mo.})^1 \\
& \quad - \frac{10,000}{1\text{mo.}} \cdot (y-2\text{mo.})^1 - \frac{60,000}{1\text{mo.}} \cdot (y-3\text{mo.})^1 - \frac{20,000}{1\text{mo.}} \cdot (y-4\text{mo.})^1
\end{align*}
\]

**Billing**

For brevity, the zero intercepts in Eqs. 4 and 5 will be omitted in further calculations. Billing equations for A and other activities plus overhead are derived by inserting the floor operator \(\lfloor \cdot \rfloor\) into Eq. 4 to give Eq. 6, which breaks the linear growth into steps.

\[
\text{Eq. 6} & \quad z(y)_{A, \text{bill}} = \frac{25,000}{1\text{mo.}} \cdot \lfloor y \rfloor - 0\text{mo.}^1 - \frac{25,000}{1\text{mo.}} \cdot \lfloor y \rfloor - 2\text{mo.}^1
\]

Adding the individual singularity functions like Eq. 6 gives the total billed costs of Eq. 7, which steps up at the end of each month. Converting Eq. 7 into stepped form with exponent \(n = 0\), shifting the cutoff \(a\) by one period from expenses that grow from the start of each period to their billing that occurs at the end of that same period, and multiplying it by \(1 + p = 1.25\) gives the billed costs with profit of Eq. 8. C that starts only at 1.5 months is automatically correctly incorporated into the subsequent bill.

\[
\text{Eq. 7} & \quad z(y)_{\text{cost+OH, billing}} = \frac{30,000}{1\text{mo.}} \cdot \lfloor y \rfloor - 0\text{mo.}^1 + \frac{20,000}{1\text{mo.}} \cdot \lfloor y \rfloor - 1\text{mo.}^1 + \frac{40,000}{1\text{mo.}} \cdot \lfloor y \rfloor - 1.5\text{mo.}^1 \\
& \quad - \frac{10,000}{1\text{mo.}} \cdot \lfloor y \rfloor - 2\text{mo.}^1 - \frac{60,000}{1\text{mo.}} \cdot \lfloor y \rfloor - 3\text{mo.}^1 - \frac{20,000}{1\text{mo.}} \cdot \lfloor y \rfloor - 4\text{mo.}^1
\]

\[
\text{Eq. 8} & \quad z(y)_{\text{cost+OH, profit}, \text{bill}} = 37,500 \cdot (y-1\text{mo.})^0 + 87,500 \cdot (y-2\text{mo.})^0 + 100,000 \cdot (y-3\text{mo.})^0 \\
& \quad + 25,000 \cdot (y-4\text{mo.})^0
\]

**Payment**

It is found that the retainage of 10% is applied until including 2 months, whereafter $125,000 of value is exceeded and the full 100% of the billed costs with profit are paid. Eqs. 9 and 10 model the factors for the payment less retainage and the retained amount that is released in the final payment and are applied to the billed costs of Eq. 7. One time period is added to the cutoff \(a\) of their second term for the billing delay \(b\).

\[
\text{Eq. 9} & \quad z(y)_\text{less retainage} = (1 - 0.1) \cdot (y-0\text{mo.})^0 + 0.1 \cdot (y-(2\text{mo.}+1\text{mo.}))^0
\]
\[ z(y)_{\text{retained amount}} = 0.1 \cdot (y - 0 \text{ mo.})^0 - 0.1 \cdot (y - (2 \text{ mo.} + 1 \text{ mo.}))^0 \quad \text{Eq. 10} \]

The multiplication of a singularity function for payments multiplied by the retainage function of Eq. 9 uses the operator \( \otimes \), which modifies the multiplication of Eq. 11 in that the result is not a single value, but a modified singularity function for payments. In other words, only half of all pairwise terms within the multiplication are executed.

\[
\left[(1-0.1) \cdot (y - 0 \text{ mo.})^0 + 0.1 \cdot (y - (2 \text{ mo.} + 1 \text{ mo.}))^0\right] \otimes \left[(1-0.1) \cdot (y - (1 \text{ mo.} + 1 \text{ mo.}))^0 + 0.1 \cdot (y - (3 \text{ mo.} + 1 \text{ mo.}))^0\right] + \left[(0.9 + 0) \cdot 37,500 \cdot (y - (4 \text{ mo.} + 1 \text{ mo.}))^0 \right] = 33,750
\]

Evaluating Eq. 11 with a regular multiplication operator after the final payment date at \( y_{\text{max}} + b + 1 \text{ mo.} = 6 \text{ mo.} \) gives \(0.9 + 0) \cdot 37,500 \cdot 1 + (0.9 + 0) \cdot 87,500 \cdot 1 + (0.9 + 0.1) \cdot 100,000 \cdot 1 + (0.9 + 0.1) \cdot 25,000 = 237,500 \) as the total payments less retainage. Since the singularity function for payments multiplied by the retained amount function of Eq. 10 is a numerical value, the multiplication operator of Eq. 12 is unmodified. This dollar value is finally released at the time \( y_{\text{end}} = y_{\text{max}} + b = 5 \text{ mo.} \). Eq. 13 finally gives the total cash flows before the terms for the interest \( i \) are added. Note that profit is left out of all of the expense terms, as it need not be financed itself and that the duration \( \varepsilon \) is added to all payment dates to facilitate calculating interest.

\[
\left[0.1 \cdot (y - 0 \text{ mo.})^0 - 0.1 \cdot (y - (2 \text{ mo.} + 1 \text{ mo.}))^0\right] \left[(1-0.1) \cdot (y - (1 \text{ mo.} + 1 \text{ mo.}))^0 + 0.1 \cdot (y - (3 \text{ mo.} + 1 \text{ mo.}))^0\right] + \left[(0.9 + 0) \cdot 37,500 \cdot (y - (4 \text{ mo.} + 1 \text{ mo.}))^0 \right] = (0.1-0)
\]

Interest
Financing interest that is added to the overdraft balance is more complex to model than continuous expenses or periodic payments due to its compounding nature, i.e. any interest draws interest itself, which causes the singularity function that includes interest to become self-referential to earlier points in time. Again the aforementioned duration \( \varepsilon \) ensures that interest is charged directly after when the overdraft balance is assessed. Since variations in the overdraft balance are caused by (but also enable) when the schedule activities occur, creating a separate closed-form equation that only modeled interest would impractically assess financing on each line item separately.
Eq. 13 gives the overdraft balances that must be financed, still excluding any charges from compound interest itself. These self-referential terms are ‘added’ to Eq. 13 to give Eq. 14 and refer to earlier terms on the time axis, leading to a ‘nested’ equation.

\[
\begin{align*}
z(y)_{\text{cash flows}} &= \frac{30,000}{1\text{mo.}} \cdot (y-0\text{mo.})^i + \frac{20,000}{1\text{mo.}} \cdot (y-1\text{mo.})^i + \frac{40,000}{1\text{mo.}} \cdot (y-1.5\text{mo.})^i - 33,750 \\
&\quad- \frac{10,000}{1\text{mo.}} \cdot (y-2\text{mo.})^i - 78,750 \cdot (y-(3\text{mo.}+\varepsilon))^0 - \frac{60,000}{1\text{mo.}} \cdot (y-3\text{mo.})^i \\
&\quad+ \frac{100,000}{1\text{mo.}} \cdot (y-(4\text{mo.}+\varepsilon))^0 - \frac{20,000}{1\text{mo.}} \cdot (y-4\text{mo.})^i - (25,000+12,500) \cdot (y-(5\text{mo.}+\varepsilon))^0 \\
&\quad+ i \left( z(y=1\text{mo.}) \cdot (y-(1\text{mo.}+\varepsilon))^0 + z(y=2\text{mo.}) \cdot (y-(2\text{mo.}+\varepsilon))^0 + z(y=3\text{mo.}) \cdot (y-(3\text{mo.}+\varepsilon))^0 \\
&\quad+ z(y=4\text{mo.}) \cdot (y-(4\text{mo.}+\varepsilon))^0 + z(y=5\text{mo.}) \cdot (y-(5\text{mo.}+\varepsilon))^0 \right) \\
\end{align*}
\]

**Results**

Evaluating Eq. 14 under consideration of its sorted terms correctly gives all end-of-month balances as per Table 2, which gives identical results to the tabular calculation of Halpin and Woodhead (1998). For simplicity, all debt balances are positive and all surpluses are negative. Additionally calculated values at \( \varepsilon \) after each end of a time period are listed underneath their preceding values for brevity. This duration \( \varepsilon \) thus distinguishes between assessing and charging interest. The maximum amount that must be financed, i.e. the credit limit up to which a bank would have to approve this contractor is $147,553 and occurs at 3 months. This value increases to $149,028.53 if the compounded interest amount of $1,475.53 is charged exactly at the ends of each month, not incrementally thereafter. Interest reduces the initial profit from $50,000 to $46,318.68 and is shown as steps in the lower half of Figure 2. Financing ends after receiving a large payment at 4 months. The credit limit is the maximum of all values.

**Table 2: Cash Flow Values (calculated from Halpin and Woodhead 1998)**

<table>
<thead>
<tr>
<th>Item</th>
<th>Timing</th>
<th>1 mo.</th>
<th>2 mo.</th>
<th>3 mo.</th>
<th>4 mo.</th>
<th>5 mo.</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direct Cost</td>
<td>On End</td>
<td>25,000</td>
<td>65,000</td>
<td>75,000</td>
<td>15,000</td>
<td>0</td>
<td>180,000</td>
</tr>
<tr>
<td>Indirect Cost</td>
<td>On End</td>
<td>5,000</td>
<td>5,000</td>
<td>5,000</td>
<td>5,000</td>
<td>0</td>
<td>20,000</td>
</tr>
<tr>
<td>Total Cost</td>
<td>On End</td>
<td>30,000</td>
<td>70,000</td>
<td>80,000</td>
<td>20,000</td>
<td>0</td>
<td>200,000</td>
</tr>
<tr>
<td>Cumulative Cost</td>
<td>On End</td>
<td>30,000</td>
<td>100,000</td>
<td>180,000</td>
<td>200,000</td>
<td>N/A</td>
<td>200,000</td>
</tr>
<tr>
<td>Plus 25% Profit</td>
<td>On End</td>
<td>7,500</td>
<td>17,500</td>
<td>20,000</td>
<td>5,000</td>
<td>0</td>
<td>50,000</td>
</tr>
<tr>
<td>Total Billing</td>
<td>On End</td>
<td>37,500</td>
<td>87,500</td>
<td>100,000</td>
<td>25,000</td>
<td>0</td>
<td>250,000</td>
</tr>
<tr>
<td>Cumulative Billing</td>
<td>On End</td>
<td>37,500</td>
<td>125,000</td>
<td>225,000</td>
<td>250,000</td>
<td>N/A</td>
<td>250,000</td>
</tr>
<tr>
<td>Less 10% Retained</td>
<td>On End</td>
<td>0</td>
<td>3,750</td>
<td>8,750</td>
<td>0</td>
<td>N/A</td>
<td>12,500</td>
</tr>
<tr>
<td>Delayed Payment</td>
<td>On End</td>
<td>0</td>
<td>33,750</td>
<td>78,750</td>
<td>100,000</td>
<td>25,000</td>
<td>+12,500</td>
</tr>
<tr>
<td>Release Retained</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>250,000</td>
</tr>
<tr>
<td>1% Period Interest</td>
<td>At ( \varepsilon )</td>
<td>300</td>
<td>1,003</td>
<td>1,476</td>
<td>903</td>
<td>0</td>
<td>N/A*</td>
</tr>
<tr>
<td>Total Cash Flow</td>
<td>On End</td>
<td>30,000</td>
<td>100,300</td>
<td>147,553</td>
<td>90,279</td>
<td>-8,819</td>
<td>N/A</td>
</tr>
<tr>
<td></td>
<td>At ( \varepsilon )</td>
<td>30,300</td>
<td>67,553</td>
<td>70,279</td>
<td>-8,819</td>
<td>-46,319</td>
<td>N/A</td>
</tr>
</tbody>
</table>

* Note that interest is compounded, i.e. draws interest itself.
Conclusions
The previous example has served to introduce the possibilities for financial modeling and analysis that singularity functions newly bring to the construction engineering and management field. The comparison of the results showed correct results for all times, while introducing clearer distinctions for minute details such as when interest is assessed versus when it is charged. Tabular computations that provided discrete values at integer times only and other previous simplified approaches that used S-curves or envelopes can be replaced by one comprehensive singularity function. Its additive nature – despite the necessity for self-referential interest in the recursive end terms – allows adding different types of costs, e.g. steps from initial mobilization, many small transactions or an overall growth function, defining specific ranges over which they apply, and converting them into regular bills including profit at periodic, user-defined intervals. Regarding inflows, the important payment terms of billing delays, owner’s retainage, and compound interest from financing have been demonstrated with singularity functions. Realistically the interest should be included in the bills to the owner. Additional terms, e.g. the fees that banks assess on unused credit were not included in the original example and have been omitted for brevity. They could theoretically be calculated with the difference of the credit limit and the total cash flow function of Eq. 14. Extending this thought, it is also possible to create target curves for cash flow that can be used in an optimization. Different levels of detail, whether within activities, e.g. to separately track materials, equipment, and labor cost, or beyond projects, e.g. to manage a portfolio of them, are also possible. Singularity functions are right-continuous and defined for all $x$-values, which makes them independent of user-selected time units. They can be implemented in computer software, e.g. Microsoft® Excel spreadsheets or the scientific analysis environment MATLAB®. They clearly separate their components as per Eq. 1, are segmental, and versatile with regard to the shapes that they can model. This makes them a suitable tool for use in construction engineering and management education to enhance the understanding of financial and engineering economics concepts. Examples of a scale and complexity like the one presented in this paper can be evaluated manually, supported by – but not dependent upon - the graphical representation of cash flows.

Future Research
The new approach for modeling intricate financial concepts with singularity functions opens various avenues for future research, including investigating conceptual details, e.g. unused credit fees and payment processing calendars, modeling different ways to assess interest exactly, e.g. based on average daily balances instead of end-of-period balances, modeling different cash flow strategies and scenarios, e.g. self-performing work versus outsourcing and the impact of different payment terms. Research also needs to investigate its user-friendliness for practice, possible conceptual limitations, and how be best integrate this cash flow model with its underlying linear schedule for an overall optimization. Other related items for future research are time-cost tradeoff to shorten project durations and progress measurement in earned value management.

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References


